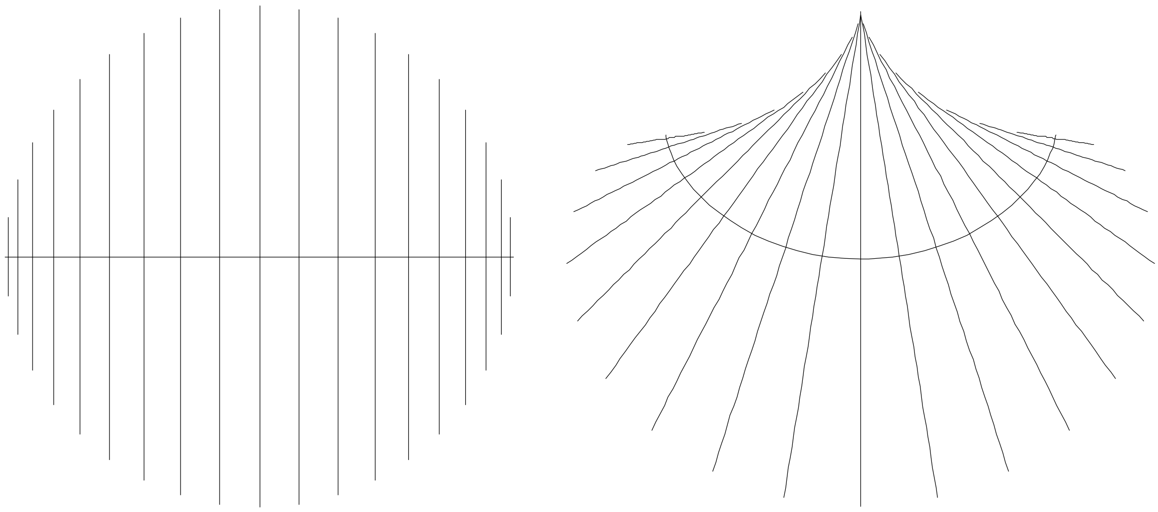
We consider the diameter of a circle and the chords perpendicular to this diameter. We want to bend the diameter to a curve such that, at each point of the bent curve, the radius of curvature is equal to the half length of the chord. Then, the bent curve is a cycloid.



We suppose that the length of the diameter is 2. Let s be the abscissa of a point M of this diameter. It's also the arc lenght of the bent curve, from the middle of the bent curve to this point M. Let R be the radius of curvature. Then R = $\sqrt{1-s^2}$, $-1 \leq s \leq 1$. But R = $\frac{\mathrm{d}s}{\mathrm{d}\alpha}$ where α is the angle between Ox and the tangent vector $\boldsymbol{T}$ to the bent curve at M. So :

$$\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \sqrt{1-s^2}$$

so $\alpha = \arcsin(s)$, $s = \sin(\alpha)$, $\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \cos(\alpha)$

But $\boldsymbol{T} = (\cos(\alpha), \sin(\alpha)) = (\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}\alpha})$ so :

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos(\alpha)$$

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = \cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$$

so $x = \frac{u+\sin(u)}{4}$ where $u = 2\alpha$

Then $\frac{\mathrm{d}y}{\mathrm{d}s} = \sin(\alpha)$

$$\frac{\mathrm{d}y}{\mathrm{d}\alpha} = \sin(\alpha)\cos(\alpha) = \frac{\sin(2\alpha)}{2}$$

so $y = \frac{1-\cos(u)}{4}$

To get the animation, we can do the same calculation with the radius of curvature equal to $\frac{\sqrt{1-s^2}}{t}$.
For $t = 0$, the radius is infinite. We have the initial diameter. For $t = 1$, we have the cycloid. If we go from $t = 0$ to $t = 1$, we go from the diameter to the cycloid.

The same calculation as before gives :

$$\begin{cases} x = \frac{1}{2(1+t)} \sin \frac{u(1+t)}{2} + \frac{1}{2(1-t)} \sin \frac{u(1-t)}{2} \\ y = \frac{1}{2(1+t)} \left(1 - \cos \frac{u(1+t)}{2}\right) - \frac{1}{2(1-t)} \left(1 - \cos \frac{u(1-t)}{2}\right) \end{cases}$$

Example with Maple :

```
with(plots):

x:=proc(t,u)
if t=1 or t=-1 then (u+sin(u))/4
else sin(u*(1+t)/2)/(2+2*t)+sin(u*(1-t)/2)/(2-2*t) fi;
end:

y:=proc(t,u)
if t=1 then (1-cos(u))/4 elif t=-1 then -(1-cos(u))/4 else
(1-cos(u*(1+t)/2))/(2+2*t)-(1-cos(u*(1-t)/2))/(2-2*t) fi;
end:

C:=u->animate([x(t,u)-z*sin(t*u/2)*cos(u/2),y(t,u)+z*cos(t*u/2)*cos(u/2), z=-1..1],t=0..1,color=black):
display({seq(S(-Pi+k*2*Pi/20),k=1..19),animate([x(t,u),y(t,u),u=-Pi..Pi],t=0..1)},
            axes=none,scaling=constrained);
```