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# On the Plane Curves whose Curvature Depends on the Distance from the Origin

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**Abstract.** Here we suggest and have exemplified a simple scheme for reconstruction of a plane curve if its curvature belongs to the class specified in the title by deriving explicit parametrization of Bernoulli's lemniscate and newly introduced co-lemniscate curve in terms of the Jacobian elliptic functions. The relation between them and with the Bernoulli elastica are clarified.

**Keywords:** classical differential geometry, plane curves, curvature, elasticity theory
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## INTRODUCTION

The most fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [1, p. 296] or [8, p. 37]). The simplicity of the situation however is quite elusive because in many cases it is impossible to find the sought-after curve explicitly. Having this in mind, it is clear that the situation is even more complicated if the curvature is given as a function of its position. Viewing the Frenet-Serret equations as a fictitious dynamical system it was proven in [10] that when the curvature is given just as a function of the distance from the origin the problem can always be reduced to quadratures. The cited result should not be considered as entirely new because Singer [9] had already shown that in some cases it is possible that such curvature gets an interpretation of a central potential in the plane and therefore the trajectories could be found by the standard procedures in classical mechanics. The approach which we will follow here, however is entirely different from the group-theoretical [10] or mechanical one [9] proposed in the above cited papers. The method is illustrated on the most natural examples in the class of curves whose curvatures depend solely on the distance from the origin. Here we consider the case in which the function in question is

$$\kappa = \sigma r, \qquad r = |\mathbf{x}| = \sqrt{x^2 + z^2} \tag{1}$$

where $x, z$ are the Cartesian coordinates in the plane $XOZ$ which have to be considered as functions of the arc-length parameter $s$, and $\sigma$ is assumed to be a positive real constant.

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## THE FRENET-SERRET EQUATIONS

If $\theta(s)$ denotes the slope of the tangent to the curve with respect to the $OX$ axis one has the following geometrical relations

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(s), \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta(s), \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \sin\theta(s) \tag{2}$$

which can be deduced also from the Frenet-Serret equations (see also Fig. 1)

$$\frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s} = \mathbf{T}(s), \qquad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} \tag{3}$$

in which $\mathbf{T}$ and $\mathbf{N}$ are respectively the tangent and the normal vector to the curve and $s$ is the natural parameter along the curve. Combining (1) and (2) produces

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(r) \tag{4}$$

which in our cases leads definitely to quite unpromising equations. We will proceed (as suggested but not pursued in [9]) by going to the co-moving frame associated with the curve

$$\mathbf{x} = \xi\mathbf{T} + \eta\mathbf{N} \tag{5}$$

and accordingly the Frenet-Serret equations (3) now read

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \dot{\xi} = \kappa\eta + 1, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}s} = \dot{\eta} = -\kappa\xi. \tag{6}$$

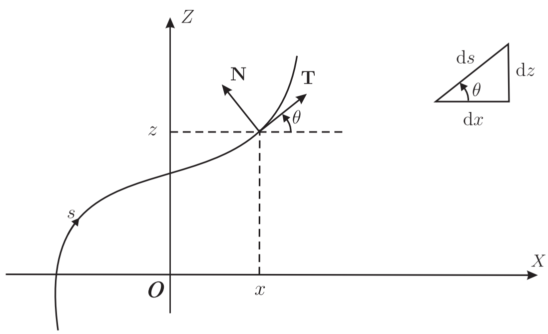


**FIGURE 1.** Geometry of the plane curve.

## INTEGRATION

Multiplying the first equation in (6) by $\xi$, the second one by $\eta$ and summing the so obtained expressions we find that

$$\xi = r\dot{r} \tag{7}$$

where the dot means a differentiation with respect to the arc-length parameter. Substituting this expression back into equation (6) and integrating we obtain

$$\eta = -\int \kappa(r) r \mathrm{d}r + c \tag{8}$$

where $c$ is the integration constant. One should notice however (cf. equation (5)) that the coordinates in the moving frame are not entirely independent but obey to the constraint

$$\xi^2 + \eta^2 = r^2 \tag{9}$$

which in view of the equations (7) and (8) presents an ordinary differential equation for the radial coordinate.

## BERNOULLI'S LEMNISCATE

This curve being a special case (when $a \equiv c$) of the Cassinian ovals [5]

$$(x^2 + z^2)^2 - 2a^2(z^2 - x^2) + a^4 - c^4 = 0 \tag{10}$$

has a curvature which is linear in $r$. Inserting $\kappa = \sigma r$ into equation (8) produces

$$\eta = -\frac{\sigma r^3}{3} \tag{11}$$

(the integration constant is taken to be zero) and the scheme from the previous section leads to the equation

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1 - \frac{\sigma^2 r^4}{9}}. \tag{12}$$

Its integration is immediate and gives

$$r = \sqrt{\frac{3}{\sigma}}\,\mathrm{cn}\left(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\right) \tag{13}$$

where $\mathrm{cn}(u,k)$ denotes one of the Jacobian elliptic functions in which the first slot is occupied by its argument and the second one by the so called elliptic modulus (a real number between zero and one). More details about elliptic functions and integrals can be found in [3] and [7]. Entering with this solution into equations (7) and (11) has as a result

$$\begin{aligned} \xi &= -\sqrt{\frac{6}{\sigma}}\mathrm{cn}\left(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\right)\mathrm{dn}\left(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\right)\mathrm{sn}\left(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\right) \\ \eta &= -\sqrt{\frac{3}{\sigma}}\mathrm{cn}^3\left(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\right). \end{aligned} \tag{14}$$

Written in terms of its components equation (5) tells us that the lemniscate coordinates $x,z$ are obtained from $\xi,\eta$ via a plane rotation specified by the slope angle $\theta$, i.e.,

$$x=\xi\cos\theta-\eta\sin\theta, \qquad z=\xi\sin\theta+\eta\cos\theta. \tag{15}$$

Obviously, what remains to be done is to find $\theta$ that can be furnish via an integration of the first equation in (2). In this way we obtain

$$\theta=3\arccos(\mathrm{dn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}}))=3\arcsin(k\,\mathrm{sn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})). \tag{16}$$

Now we have to take into account the trigonometric identities

$$\sin 3\varphi=3\sin\varphi-4\sin^3\varphi, \qquad \cos 3\varphi=4\cos^3\varphi-3\cos\varphi \tag{17}$$

which give

$$\begin{aligned}\sin\theta &= \frac{3}{\sqrt{2}}\mathrm{sn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})-\sqrt{2}\,\mathrm{sn}^3(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})\\ \cos\theta &= 4\mathrm{dn}^3(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})-3\mathrm{dn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})\end{aligned} \tag{18}$$

and eventually provide the parametrization of the Bernoullian lemniscate. By making repeating use of the fundamental identities, which the Jacobian elliptic functions $\mathrm{sn}(u,k),\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ obey, i.e.,

$$\mathrm{sn}^2(u,k)+\mathrm{cn}^2(u,k)=1, \qquad \mathrm{dn}^2(u,k)+k^2\,\mathrm{sn}^2(u,k)=1 \tag{19}$$

it is possible to simplify the expressions for $x$ and $z$ into the form

$$\begin{aligned}x &= \sqrt{\frac{3}{2\sigma}}\mathrm{cn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})\mathrm{sn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})\\ z &= -\sqrt{\frac{3}{\sigma}}\mathrm{cn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}})\mathrm{dn}(\sqrt{\frac{2\sigma}{3}}s,\frac{1}{\sqrt{2}}).\end{aligned} \tag{20}$$

The properties of the Jacobian functions make obvious also the relations

$$\eta^2=\left(\frac{\sigma}{3}\right)^2(\xi^2+\eta^2)^3, \qquad z^2-x^2=\frac{\sigma}{3}\left(x^2+z^2\right)^2 \tag{21}$$

and the last one is just the standard form of the Bernoulli's lemniscate (cf. equation (10)). In polar coordinates $\xi=r\cos\psi$, $\eta=r\sin\psi$, $x=r\cos\phi$, $z=r\sin\phi$ the above algebraic curves of degree six, respectively four, take the forms

$$\sin\psi=-\frac{\sigma}{3}r^2, \qquad \cos 2\phi=-\frac{\sigma}{3}r^2. \tag{22}$$

This remarkable similarity of their polar representation suggests to calculate the curvature of the co-lemniscate. The most convenient method in this situation seems to be the application of the formula

$$\kappa = \frac{|r^2 + 2\dot{r}^2 - r\ddot{r}|}{(r^2 + \dot{r}^2)^{3/2}} \tag{23}$$

in which this time the dots denote differentiations with respect to the polar angle. By straightforward but tedious calculations one ends with the formula

$$\kappa = \frac{2\sigma r\left(\sigma^2 r^4 + 9\right)}{\sqrt{3}\left(\sigma^2 r^4 + 3\right)^{3/2}}. \tag{24}$$

Regardless of how similar to the lemniscate seems to be this curve in the polar coordinates its curvature is quite different from that of the parent curve. Both curves are plotted for illustration in Fig. 2.

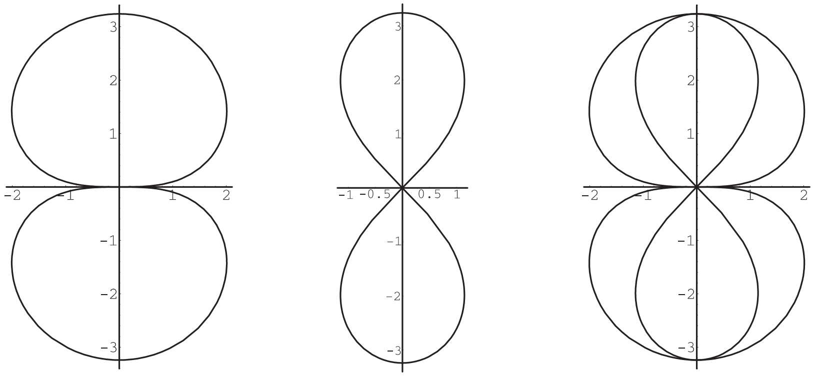


**FIGURE 2.** The Bernoullian co-lemniscate (left), Bernoulli's lemniscate (middle) and both of them (right) drawn via formulas (14) and (20) with $\sigma = 3.5$.

**Remark.** Because of the relation between the radial coordinate $r$ and the curvature equation (12) can be rewritten into the form

$$\dot{\kappa}^2 + \frac{\kappa^4}{9} = \sigma^2 \tag{25}$$

which can be recognized and referred further on as the *intrinsic equation* of the Bernoulli's lemniscate.

## THE LEMNISCATE AND BERNOULLI'S ELASTICA

We will end this paper by outlining the relation of the Bernoulli lemniscate to another famous curve invented by Bernoulli - the so called free (or rectangular) elastica [2] which appears also as a profile curve of the Mylar balloon [6].

For that purpose, let us differentiate (25) which gives

$$\ddot{\kappa}_{\text{lemn}} + \frac{2}{9}\kappa_{\text{lemn}}^3 = 0 \tag{26}$$

and presents another form of the intrinsic equation of the lemniscate. By comparing it with the intrinsic equation of the free elastica

$$\ddot{\kappa}_{\text{elas}} + \frac{1}{2}\kappa_{\text{elas}}^3 = 0 \tag{27}$$

it is easy to conclude that one can pass from one to the other by the following transformation

$$\kappa_{\text{lemn}} = \frac{3}{2}\kappa_{\text{elas}} \tag{28}$$

which has been noticed also recently by Matsutani [4]. Actually, the mathematical reason is that the curvature of the Bernoulli elastica is also a linear function of the distance but in this case - from the $OX$ axis.

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The first author would like to express his thanks to Professor S. Woronowicz who during the conference talk asked the question which is the curve whose curvature is $\sigma/r$?
The answer turns out to be that when the particular value of the parameter $\sigma = 1$ the curve is known as Norwich or Sturmian spiral [11, pp. 139-140]. For $1 < \sigma < \frac{2}{\sqrt{3}}$ the generalized Sturmian spirals resembles the ordinary ones. When $\sigma > \frac{2}{\sqrt{3}}$ the curve is finite and bounded and lies between two circles centered at the origin. For $\sigma = \frac{2}{\sqrt{3}}$ the polar angle between the initial and the end point is exactly $\pi$ and for $\sigma > \frac{2}{\sqrt{3}}$ it is given by the formula $\frac{\sigma\pi}{\sqrt{\sigma^2-1}}$. Due to the lack of space we cannot provide the full details here but we hope to be able to report on this subject soon elsewhere.

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