16
17

Plane mechanisms and the "downhill principle"

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Abstract — **PRELIMINARY – comments desirable. We survey and redo the whole mathematical area of "planar mechanisms:" machines made of rigid parts constrained to lie in a plane and linked by rotatable rivets. This area attracted great interest in England and France in the late 1800s, and later enjoyed a brief revival in Russia in the 1940s, but has largely lain dormant. Important contributions of the present work include:**

1. **A streamlined algebraic method is proposed to analyse simple planar mechanisms, allowing the first analysis of the 5-link anchored mechanisms.**
2. **The "algebraic completeness theorem" for planar mechanisms is revisited. Algebraic completeness theorems have attracted a lot of interest in the last decade, but none of the workers realized that the first such result had been shown about a century earlier!**
3. **We show how to enumerate all combinatorial types of planar mechanisms with N links, and provide asymptotic bounds and a table of counts for small N computed by Henry Cejtin.**
4. **Assuming the "downhill principle" that physical systems not at local minima of the potential energy function always go downhill, would allow planar mechanisms to solve NP-hard problems quickly. But we show that in both classical and quantum mechanics, the downhill principle cannot be relied upon.**

Keywords — Downhill principle, algebraic completeness, planar mechanisms, isomorph-free generation of combinatorial objects, graphs, hypergraphs, 3-bar curve, conicographs, line drawing devices, NP-completeness, Church's thesis, quantum mechanics, classical mechanics.

Contents

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1 Introduction

This paper has two parts. In the first part, we re-examine the mathematics of 2D mechanisms made of rigid parts. In the second part, we consider the "downhill principle," and computational devices based on it.

1.1 2D mechanisms — summary

Rigid bars constrained to a plane and joined at their ends by rivets, are capable of computing any algebraic function. This was essentially known to A.B.Kempe in 1877, with other authors providing more complete proofs during the period 1877-1902. This predates the large body of modern work, beginning in 1985, on "algebraic universality." Hence (as we'll see) the question of whether a specified "2D mechanism" is realizable is ETR-complete; the question of whether it can be moved from one configuration to another (the "algebraic connectedness problem") is PSPACE-complete. We'll also see that the "trigonometric connectedness problem," arising by also allowing mechanisms featuring circular wheels rolling along lines, is undecidable.

We'll re-prove the algebraic universality result completely and elegantly, and also extend it to show that *planar* mechanisms (with no crossings) suffice. We also re-examine and re-work the whole area of 2D mechanisms, presenting a new, more powerful method for studying their algebraic formulae and consequently the first analysis of the 5-bar anchored linkages, a better analysis of the 3-bar ones, upper and lower bounds on the degree of the curves traced by mechanisms, a technique for systematically enumerating the combinatorial types of mechanisms, and tables of (and asymptotic bounds on) their counts.

1.2 Downhill principle and computational devices based on it — summary

Given a multivariate polynomial of bounded degree and with integral coefficients expressed in unary, it is NP-complete to decide whether a specified point X is a local minimum (§11). This is true even if we promise that all minima are at least 1 below, and at least distance 1 from, X. But we can set up a physical system, in fact a planar mechanism, whose potential energy function is any desired such polynomial of the system's degrees of freedom. If one believes that the laws of physics imply the appealing "downhill principle" that any physical system will go downhill in potential energy if it is not at a local min, then we can set up a physical system to quickly solve NP-hard problems. This was first realized by Vergis et al. [93] in 1986.

But, we argue, with Newton's classical laws of motion and assuming the fundamental potentials of physics (e.g. Coulomb's) have a continuous second derivative (which they do), the downhill principle is *not* valid. Indeed, we show that in *any* physical situation in which it is NP-hard to tell if we are at a local min, we must be at a stationary point of at least 3rd order – and instabilities of this high an order are too weak to force finite progress in finite time. We also point out that there exist Hamiltonian systems in which a nonzero measure set of orbits exist, which never leave the neighborhood of a non-minimal point. This is a second refutation of the downhill principle in noiseless classical physics.

In noiseless quantum physics, we show that the downhill principle is simultaneously both more than true – in the sense that tunneling is possible, and indeed cannot be reduced to zero although it may be made exponentially small – and false – in the sense that we can set up a smooth, well behaved potential function with no basins anywhere, and in which a path monotonically decreasing to $-\infty$ exists starting from anywhere, but in which the spectrum is discrete and in which almost all the probability mass is confined within a bounded region.

In the presence of sufficiently small amounts of thermal noise, in one model the downhill principle in classical physics seems to be valid, but it can still take an exponentially long time to make discernible progress, but in a more realistic model – in which the "thermal noise" really is caused by interaction with a larger (also noiseless) system – again the downhill principle is not necessarily valid.

2 2D mechanisms – basic definitions and notation

A "2D mechanism" consists of a number of rigid line segments ("bars") in the plane, possibly attached at their endpoints by "rivets." The bars can move freely and the rivets can rotate freely, provided everything remains in the plane and the attachments at rivets are not broken.

When designing mechanisms, it helps to adopt some simplifying notation.

(i) We may, without loss of generality, regard the rivets to be of two kinds: "free," and "nailed to the wall." This is because, since triangles are rigid, we could replace the "wall" by a triangulated framework whose vertices included all the "nailed" rivets. We will therefore draw nailed rivets with *two* concentric circles.

(ii) We may also, without loss of generality[1] allow rivets to be located in the middle, as well as at the endpoints, of line segments. Again, this is because we could implement each bar with N rivets in the middle, by a triangulated truss made of $2N+5$ bars. Indeed, for the same reason we may allow the bars to be rigid polygons, rather than line segments. We'll fill such polygons with grey shading as in

[1]Unless we are demanding true planarity, i.e. disallowing crossings or overlap of any two bars, unless one of them is the wall. In this case, we would disallow two bars from being linked at a rivet which is at neither bar's endpoint.

figure 1. (Indeed, many people regard the number of *bars* as irrelevant and consider rigid *polygons* to be fundamental – a view we'll take in §8.)

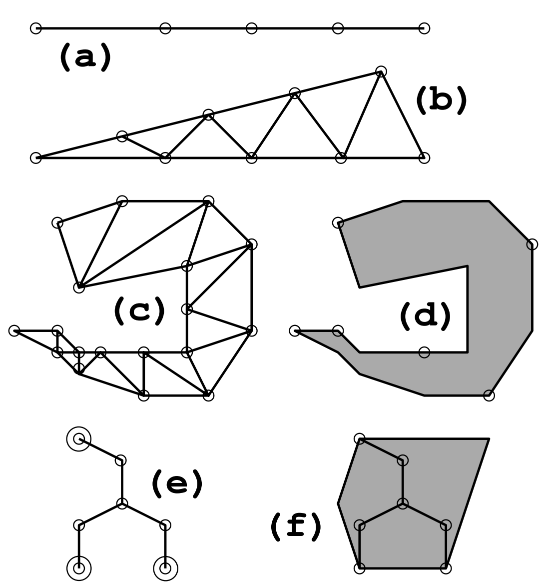

Figure 1: 2D mechanism notation. (a) "Bar with N rivets" ($N-2$ of them not at its endpoints) may be implemented as (b) triangulated truss with $2N+5$ bars. Rigid N-gons (d) may be implemented as triangulated trusses with $2N-3$ bars (c). Anchored mechanisms (e) may be implemented with an additional rigid polygon serving as the "wall" (f).

We will sometimes demand "planarity," i.e. demand that no two bars of an anchored mechanism are allowed to cross.

The "combinatorial type" of a 2D mechanism is a hypergraph [7] whose vertices are the rivets and whose hyperedges are the (rigid polygonal) links, i.e. the specification of which bars are joined to which rivets. But as we'll see in §8, in the "generic" case, hypergraphs are not really needed because their duals are old fashioned graphs.

Points on mechanisms with one degree of freedom follow an algebraic curve trajectory. It is possible for such curves to self-intersect, i.e. cross, in which case we can have a "dead point" at which the mechanism can choose to follow one or the other branch of the curve. Dead points are important for two reasons: (1) In many applications it is important to prevent the mechanism from making the wrong choice! (2) Dead points often involve zero or infinite mechanical advantage. Although discussion of dynamics (e.g. forces, velocities, accelerations) is beyond the scope of this paper, still, we can see the desirability of knowing about – and usually of avoiding – dead points.

The final thing it is important to know about is the "range of motion" of a mechanism. Any mechanism of the type we are considering (finite length rods) obviously has only a finite range of motion and will only trace out algebraic curve *segments*, and it is desirable to know the endpoints of these segments.

3 The lore of flexible quadrilaterals

The sextic curve traced by a point on the middle bar of a 3 bar anchored linkage (figure 2) has been discussed by many previous authors, the first of whom were Roberts[2] [80] and Johnson[3] [45].

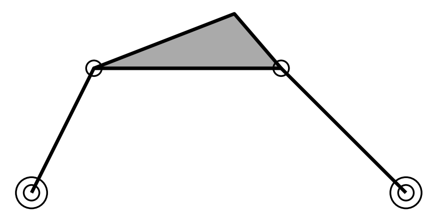

Figure 2: The 3 bar anchored linkage – the simplest nontrivial mechanism.

With the benefit of hindsight, it is possible to simplify, symmetrize, and clarify their analyses a lot. There have also been analyses depending heavily on the properties of elliptic functions [23] [67] [6], but we won't discuss these since the ideas in those papers seem to work *only* for the 3-bar linkage and won't generalize.

Johnson's work involved a slick application of the fact (derivable by use of the cosine addition formula) that if $\alpha+\beta+\chi=n\pi$ where n is an integer, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\chi - 2(-1)^n\cos\alpha\,\cos\beta\,\cos\chi - 1 = 0. \quad (1)$$

However, Johnson could only derive a formula for the curve traced by a point on the middle bar *line segment* of a three bar linkage. Roberts worked out the fully general case, allowing the middle bar to be a general *triangle*, with the "pen" lying at its third vertex.

After a sketchy derivation of an equation obeyed by this curve (contaminated by a typo) Roberts somehow noticed[4] buried in his algebra the following remarkable theorem:

Theorem 1 (Roberts's triple generation theorem) *Any three bar curve may be generated by three different 3-bar mechanisms, whose axes are the endpoints of each of the three sides of the "focal triangle" (a triangle similar to the traversing traingle whose base had the two original anchored axes as endpoints; see figure 3).*

[2] Samuel Roberts (1827-1913) was president of the London Math. Society 1880-1882. His work was mainly in geometry and number theory. An obituary is Proc. Royal Soc. of London 89 A (1913-1914) xx-xxi.

[3] William Woolsey Johnson (1841-1927), one of the foremost American mathematical textbook writers of his generation. Johnson graduated from Yale in 1862 and for most of the rest of his life taught at the US Naval academy in Annapolis MD. He was a founding member of the American Math. Society. Johnson authors at least 7 textbooks, most published by J.Wiley & Sons NY: "Curve tracing" (1884), "An elementary treatise on analytical geometry," "An elementary treatise on integral calculus," "An elementary treatise on theoretical mechanics" (2 vols in 1; 1901) "Differential calculus" (1904 revision of an 1879 edition he'd coauthored with J.M.Rice) "Treatise on ordinary and partial differential equations" (1910), and "Errors and least squares" (1915). Obituary: NY Times 16 May 1927.

[4] Chebyshev also rediscovered triple generation.

A simple explanation for this was later found by Cayley [15] and Clifford [19], see figure 3.

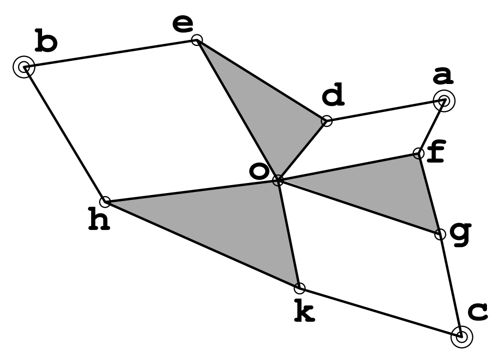

Figure 3: Roberts's triple generation, as explained by Clifford. Let $\triangle ohk \sim \triangle deo \sim \triangle fog$ and let $oghc$, $ohbe$, $adof$ be parallelograms. Claim: no matter how the un-anchored mechanism is distorted, $\triangle abc$ will remain similar to $\triangle ohk$. [Proof: the operation of turning through the angle $\angle hok$ and scaling the length by $\overline{oh}/\overline{ok}$ converts $oh \to ok$, $eb \to gc$, $de \to do$, $de \to do =_H af$, $ad =_H fo \to fg$, where "$=_H$" means "parallel and congruent to." Hence the whole path ab (which may be regarded as a composite of ad, de, eb) is converted into ac. Thus $\triangle abc \sim \triangle ohk$, Q.E.D.] Hence if we now anchor any two of $\{a, b, c\}$ then the third will automatically be anchored. But the mechanism remains flexible! Conclusion: a pen attached to the point o on any of the anchored 3-bar submechanisms $adeb$, $bhkc$, and $cgfa$ will each trace the same curve.

By combining Johnson's idea with the Cayley-Clifford explanation of Roberts's triple generation theorem, we can get a very slick, new derivation of a pleasantly symmetrical form of Roberts's equation. Refer to figure 4 for the notation.

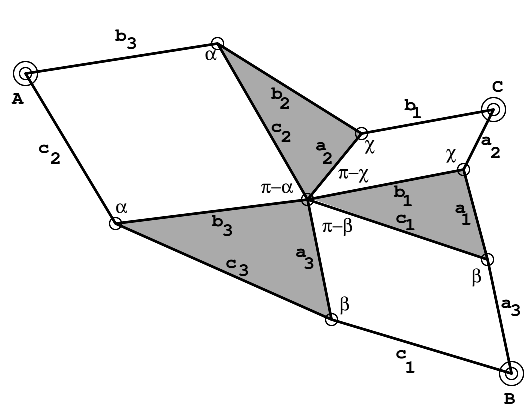

Figure 4: Notation for new derivation of 3-bar curve equation. The "pen" (point being traced) is the central vertex.

Theorem 2 (The 3-bar curve) *In a 3-bar linkage with anchored axis A attached to a crank of radius c_2 jointed to a rigid "traversing triangle" with sides a_3, b_3, c_3, jointed to a crank of radius c_1 anchored at B (lower part of figure 4): the third vertex z of the triangle obeys*

$$\begin{aligned} &a_1^2(b_3^2 + c_2^2 - |z-A|^2)^2 + \\ &b_2^2(a_3^2 + c_1^2 - |z-B|^2)^2 + \\ &c_3^2(a_2^2 + b_1^2 - |z-C|^2)^2 + \\ &\quad (b_3^2 + c_2^2 - |z-A|^2) \\ &\quad (a_3^2 + c_1^2 - |z-B|^2) \\ &\quad (a_2^2 + b_1^2 - |z-C|^2) \\ &\quad -4a_1^2 b_2^2 c_3^2 \;=\; 0. \end{aligned} \tag{2}$$

Here z, A, B, C are regarded as complex numbers, i.e. $z = x + iy$. C is the point such that $\triangle ABC$ is similar to, and chirally the same as, the traversing triangle, and $a_1 = a_3 c_1/c_3$, $a_2 = a_3 c_2/c_3$, $b_1 = b_3 c_1/c_3$, $b_2 = b_3 c_2/c_3$.

Proof.[5]
Because the sum of the angles of a triangle is π, one sees by considering the 6 angles around z that the 3 grey angles in figure 4 sum to π, hence also $(\pi-\alpha)+(\pi-\beta)+(\pi-\chi) = \pi$, hence $\alpha + \beta + \chi = 2\pi$. By the law of cosines

$$\cos\alpha = \frac{b_3^2 + c_2^2 - |z-A|^2}{2b_3 c_2} \tag{3}$$

$$\cos\beta = \frac{a_3^2 + c_1^2 - |z-B|^2}{2a_3 c_1} \tag{4}$$

$$\cos\chi = \frac{a_2^2 + b_1^2 - |z-C|^2}{2a_2 b_1} \tag{5}$$

and now by substituting these into the case $n = 2$ of (EQ 1) times $4a_2^2 b_1^2 c_3^2$ and using $a_3 = a_1 c_3/c_1$, $a_2 = a_1 c_2/c_1$, $b_1 = b_2 c_1/c_2$, $b_3 = b_2 c_3/c_2$, we get (EQ 2). ☐

Theorem 3 (Johnson's form for a subset of 3-bar curves) *Let the two anchored axes of a 3-bar linkage be at ± 1; let the three bar lengths be a, b, c; and let the pen be located a fraction f along the traversing bar (of length b), as in figure 5. Define $g = 1 - f$. Then the equation of the curve z traced by the pen is*

$$\begin{aligned} g^2|z-1|^2(|z+1|^2 + f^2b^2 - a^2)^2 + 4g^2f^2b^2(|z|^2-1)^2 + \\ f^2|z+1|^2(|z-1|^2 + g^2b^2 - c^2)^2 \\ +2fg(|z+1|^2 + f^2b^2 - a^2)(|z|^2-1)(|z-1|^2 + g^2b^2 - c^2) \\ -4g^2f^2b^2|z^2-1|^2 = 0. \end{aligned} \tag{6}$$

Proof. As usual we express x, y coordinates using a complex number $z = x + iy$.

We may solve for $\cos\alpha$, $\cos\beta$, and $\cos\chi$ in figure 5 by using the law of cosines. Define $g = 1 - f$ for compactness. Then

$$\cos\alpha = \frac{f^2b^2 + |z+1|^2 - a^2}{2fb|z+1|} \tag{7}$$

$$\cos\beta = \frac{|z+1|^2 + |z-1|^2 - 2^2}{2|z+1| \cdot |z-1|} = \frac{|z|^2 - 1}{|z^2 - 1|} \tag{8}$$

[5] A formula resembling (EQ 2), but apparently incorrect, was given without proof by Hippisley [39]. In fact all of [39]'s section 1 seems to be somewhere between extremely vague and incorrect or unjustified.

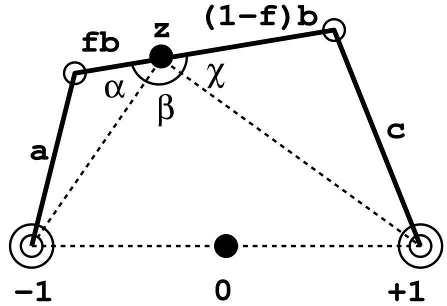

Figure 5: Notation used for Johnson's 3-bar curve formula.

$$\cos\chi = \frac{g^2b^2 + |z-1|^2 - c^2}{2gb|z-1|}. \qquad (9)$$

Hence by combining the previous three equations by using the case $n = 1$ of (EQ 1) we see that the equation of the curve traced by z as the mechanism moves is

$$\left(\frac{|z+1|^2 + f^2b^2 - a^2}{2fb|z+1|}\right)^2 + \left(\frac{|z|^2-1}{|z^2-1|}\right)^2 + \left(\frac{|z-1|^2 + g^2b^2 - c^2}{2gb|z-1|}\right)^2$$
$$+2\frac{|z+1|^2 + f^2b^2 - a^2}{2fb|z+1|}\,\frac{|z|^2-1}{|z^2-1|}\,\frac{|z-1|^2 + g^2b^2 - c^2}{2gb|z-1|} - 1 = 0. \qquad (10)$$

To convert this equation into sextic-polynomial$(x, y) = 0$, we multiply it by $4f^2g^2b^2|z^2-1|^2$ to get (EQ 6). (This all remains valid if the links form a crossed quadrilateral. In that case, e.g., α becomes negative, that is all.) □

Johnson [44] also derived a formula for the equation obeyed by the lengths of the diagonals of flexible quadrilaterals. Unfortunately his formula had a couple of typos and was only applicable to the subclass of quadrilaterals with $a^2 + c^2 = b^2 + d^2$, in which the diagonals remain at constant angle (indeed a right angle) during flexure[6]. A corrected version of his formula, and now generalized to be valid for all flexible quadrilaterals, was found by Bowman [9] 75 years later.

Theorem 4 (Bowman's diagonals theorem) *Let p and q be the diagonals of a flexible quadrilateral with sides a, b, c, d in cyclic order. Then*

$$(p^2 + q^2 - [a^2 + b^2 + c^2 + d^2])p^2q^2$$
$$-(a^2 - d^2)(c^2 - b^2)p^2 - (b^2 - a^2)(d^2 - c^2)q^2$$
$$+(b^2d^2 - a^2c^2)(b^2 + d^2 - a^2 - c^2) = 0. \qquad (11)$$

(New, simple) Proof. By the law of cosines $a^2+b^2-p^2 = 2ab\cos\theta$ and $a^2 + d^2 - q^2 = 2ad\cos\psi$ where θ is the angle between the sides a and b, and ψ between sides a and d. Now $(a - b\cos\theta - d\cos\psi)^2 + (d\sin\psi - b\sin\theta)^2 = c^2$, i.e., $(a - b\cos\theta - d\cos\psi)^2 + d^2 + b^2 - d^2\cos^2\psi - b^2\cos^2\theta - c^2 = 2bd(\sin\theta)(\sin\psi)$. Now squaring, using $\cos^2 + \sin^2 = 1$ to get rid off all sin's, substituting in the values of $\cos\theta$ and $\cos\psi$, multiplying by a^2, and expanding leads to (EQ 11). □

Bowman's equation is of fundamental importance because if the squared length of one diagonal is regarded as known, the other may be found by solving a quadratic equation[7]. Then $\cos\theta$, $\cos\psi$ are deducible by the law of cosines, $\sin\theta$, $\sin\psi$ are deducible by $\cos^2 + \sin^2 = 1$, and then all the coordinates of everything are deducible, all without needing any transcendental functions nor even any algebraic ones beyond square root.

3.1 Simpler cases of the 3-bar curve

The curve (EQ 2) is generically of degree 6. However, with special choices of the parameters a_3, b_3, c_3, c_2, c_1, it could factor into curves of lower degree.

Donelan [25] found a complete characterization of the parameter choices leading to "nongenericity," but Donelan's definition of "generic" concerns the topological structure of the algebraic curve and its singularities, and it is not obvious that it necessarily has anything to do with factorizability or some other natural[8] notions of nongenericity. But in fact, the cases in which the sextic (EQ 2) factors are precisely Donelan's non-generic cases, as was shown by Gibson and Newstead [31].

Theorem 5 (Gibson-Newstead non-genericity theorem) *In the notation of figure 4, the 3-bar sextic (EQ 2) factors if and only if*

$$\pm c_2 \pm c_3 \pm c_1 = \overline{AB}. \qquad (12)$$

Going through these 8 cases (really fewer when the symmetry exchanging c_1 and c_2 is taken into account) one by one, we find:

1. $c_1 + c_2 + c_3 + \overline{AB} = 0$: Unrealizable.
2. $c_1+c_2+c_3 = \overline{AB}$ or $c_1+c_2+\overline{AB} = c_3$ or $c_1+c_3+\overline{AB} = c_2$ or $c_2 + c_3 + \overline{AB} = c_1$: Rigid rod.
3. $c_1 + c_2 = c_3 + \overline{AB}$: Sums of opposite sidelengths are equal.
4. $c_1 + c_3 = c_2 + \overline{AB}$: Sums of adjacent sidelengths are equal.
5. Subcase of the previous case: $c_1 = c_2$, $c_3 = \overline{AB}$: Parallelogram or crossed parallelogram. In the uncrossed case, the curves are circles (degree 2) whereas in the crossed case they are curves of degree 4.

[6] In this case, (EQ 11) is

$$(p^2 + q^2 - 2[a^2 + c^2])p^2q^2 + (a^2 - d^2)^2p^2 + (b^2 - a^2)^2q^2 = 0.$$

[7] Alternatively, one could regard pq or p^2+q^2 as the initially known parameter and solve quadratics to get p and q. This might yield more pleasingly symmetrical expressions. The point is that in all cases we get a parametric description of the curve traced by the apparatus.

[8] For example, as we mentioned before, the quadrilaterals with $a^2 + c^2 = b^2 + d^2$ are precisely the ones whose diagonals remain at constant angle (indeed, a right angle) during flexure. This seems to be a highly natural and nongeneric property, but Donelan's criterion doesn't find it.

6. Intersection of the previous 2 cases: $c_2 = c_3$, $c_1 = \overline{AB}$: Kite.

7. Further intersection: $c_1 = c_2 = c_3 = \overline{AB}$: Rhombus. Factors into 3 circles.

3.2 *Preventing parallelograms from crossing, uncrossing, and folding*

When a flexible quadrilateral is flexed until two adjacent sides become collinear, we reach a "dead point" at which the two collinear sides could bend "either way," resulting in following one of two different branches of the algebraic curve.

Consider the important example of a flexible parallelogram. When it is flexed until it flattens (all vertices collinear) we may choose to continue the motion by becoming either a parallelogram again, or a "crossed parallelogram." If the parallelogram is in fact a rhombus, then folding is also possible.

If crossing and folding of a parallelogram are considered undesirable, one may prevent them by means of one extra bar. On the other hand, if the crossed form is wanted and conversion to the uncrossed form is undesirable, then a simple means of preventing uncrossing[9] and folding is to add two extra bars. These are illustrated in figure 6.

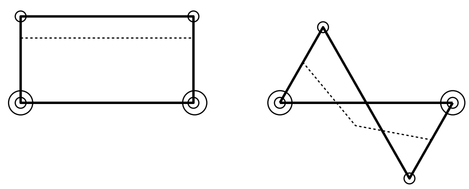

Figure 6: Parallelogram (at left). The conversion to the crossed parallelogram (at right) may be prevented by adding one extra bar (shown dashed at left). Uncrossing and flattening of the crossed parallelogram are both prevented with 2 auxilary bars (shown dashed at right) whose summed length is slightly less than the long parallelogram side.

3.3 *Approximate lines, parabolas, and circles from 3-bar linkages*

Renowned engineer and steam engine developer James Watt needed a mechanism to convert straight line reciprocating motion to circular motion, and it had to be constructible to high precision, economically, using the very low tech machining capabilities available in 1784. Was there a suitable planar mechanism?

Watt was unable to solve this problem, but he did manage to devise an anchored 3-bar mechanism for drawing an algebraic curve whose departure from straightness was of the 5th order. P.L.Chebyshev (1821-1894) also worked on this problem, producing in 1868 a better anchored 3-bar approximate line mechanism (6th order – this is best possible since the 3-bar curve is a sextic) than Watt's, and ultimately producing his well known theory of functional approximation.

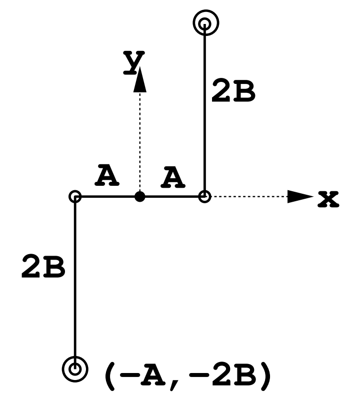

Figure 7: Approximate straight line mechanisms.

For many purposes these approximate line drawing devices are better than the exact line drawing devices discovered later, since the only have 3 links, whereas the exact line drawers require ≥ 5 links, and the approximations are very good. Due to their importance and the fact that many available analyses of these mechanisms are comparatively inaccessible, complicated[10] and/or incorrect, we redo those works here.

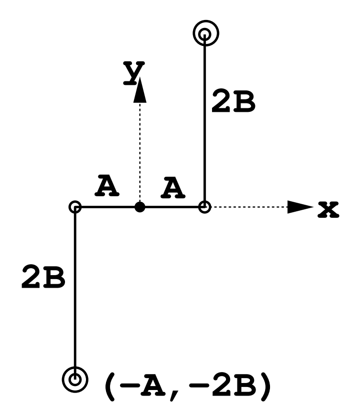

Figure 8: Watt's mechanism in central position (the angles are 90°).

Theorem 6 (Watt's curves) *The curves traced by the middle point of Watt's mechanism (figure 8) obey the equations*

$$+4B^2A^2y^2+y^4A^2+y^6+8By^3Ax-2A^2x^2y^2+2y^4x^2+x^4A^2+y^2x^4 = 0 \tag{13}$$

$$+y^6 - 8ByA^3x + 8By^3Ax + 4A^4x^2 + 3y^4x^2 - 4B^2x^2A^2 - 4A^2x^2y^2 + 4B^2x^2y^2 + 8BAyx^3 - 4x^4A^2 + 4B^2x^4 + 3y^2x^4 + x^6 = 0 \tag{14}$$

[9] But this also prevents flattening, restricting the range of rotation of one of the angles θ to, e.g. $1° < \theta < 179°$.

[10] Chebyshev was ignorant of the method of resultants [30], which would have simplified many of his analyses tremendously.

Proof. We may write the equations

$$(y - Bq + B)^2 + (x - Bp - A)^2 - A^2 = 0,$$
$$(y + Bq - B)^2 + (x + Bp + A)^2 - A^2 = 0 \qquad (15)$$

where $p^2 + q^2 = 1$. Then by taking the resultant [30] of the first equation with $p^2 + q^2 - 1$ to eliminate p, and of the second equation with $p^2 + q^2 - 1$ to eliminate p, and taking the resultant of the results to eliminate q, we get an polynomial equation of the form $P(x, y) = 0$ relating x and y. Dividing this by $256B^4$ and factoring the polynomial, the two factors are found to be (EQ 13) and (EQ 14). □

The behavior of (EQ 14) for x, y small is $x = y^5/(8BA^3) + O(y^7)$, which is indeed straight to 5th order. Unfortunately the other curve (EQ 13) is a much worse approximant and also includes the origin, leading to the risk of accidentally getting on the wrong branch.

PUT MORE HERE ON CHEBYSHEV & RELATED MECHANISMS???

4 A method for finding the algebraic curve traced by a mechanism

Our later algebraic completeness theorem 13 indicates that analysis of the curves traced by planar mechanisms is in general extremely hard – as hard as analysing any algebraic system. But that does not mean that we should entirely give up hope, especially for small problems ($\leq$ 10 links). We'll now describe a general method of attack specifically designed to make the analysis of small problems feasible. It is just a combination of a few tricks (no one of which is entirely original) whose cumulative effect is very powerful. Indeed, we'll use this method to produce the first analysis of the 5-bar anchored mechanisms (working by hand); and with computerization, similar analyses might become feasible in some cases with as many as 10 links. Numerical, rather than symbolic, methods are presumably even more powerful (although less rigorous, and they depend on having a starting point), but we won't discuss them either since they are so simple to create. Another general method has been discussed by Gibson & Newstead [31], but it seems to be much less practical than ours because, while we use a trick to decrease the number of (real) dimensions by a factor of 2, [31] instead begin by intentionally *increasing* the number of dimensions by a factor of 2. This may help them to make certain statements, but seems unhelpful computationally.

Regard the mechanism as a hypergraph, in which every rigid link is a "hyperedge," and the rivets linking them are "vertices." (We'll see in §8 that the dual of this hypergraph is an ordinary graph, at least in the generic case in which each rivet joins exactly two rigid links.)

Step 1. Find a hypertree[11] inside the hypergraph that includes the anchored hyperedge and such that every hyperedge includes at least two hypertree vertices. Preferably, choose such a hypertree with the minimum possible number H of hyperedges. (In the generic case, this corresponds in the dual graph G to finding a tree, containing the root vertex, and preferably containing the minimum possible number H of vertices, such that every vertex of G is either a tree vertex, or adjacent to at least two tree vertices.)

Step 2. Regard the unit norm complex numbers z_j, $j = 1, ..., H - 1$, determining the orientation of hypertree hyperedge j, as the fundamental variables.

Note: the coordinates of any hypertree rivet are easily expressed in terms of these fundamental variables by finding the path P in the hypertree from some fixed rivet [WLOG the origin] on the anchored hyperedge to that rivet, and adding up the corresponding displacements $\sum_{jqr \in P} c_{qjr} z_j$ where c_{qjr} are appropriate complex constants depending on the shape and size of hyperedge j and the rivets q and r on that hyperedge that are part of the path. Specifically c_{qjr} is the distance between the two rivets times $e^{i\theta}$, where θ is the counterclockwise angle from hyperedge j's orientation arrow to the direction from rivet q to rivet r. The coordinates of any point are then easily determined from the rivet coordinates.

Instead of regarding these $H - 1$ complex numbers z_j as the unknowns (which would be $2H - 2$ real unknowns), regard $H - 1$ *real* numbers t_j as the unknowns, where

$$z_j = \frac{1 + it_j}{t_j + i}. \qquad (16)$$

Then, $|z_j| = 1$ automatically. This trick immediately halves the number of real unknowns.

Step 3. We now may write down the equations satisfied by the t_j. Let u and w be any two distinct rivets on the same non-hypertree hyperedge h of the mechanism such that the edge uw is part of some predetermined rigid triangulation of the rivets of h. (For each hyperedge with n rivets we must have predetermined some $(2n - 3)$-edge triangulation.) For each such pair uw, there will be an equation. Let P_{uw} be the path in the hypertree from u to w. Then

$$\left| \sum_{jqr \in P} c_{qjr} z_j \right|^2 = |c_{uhw}|^2. \qquad (17)$$

I.e., rewriting as a polynomial equation in the t_j:

$$\left| \sum_{jqr \in P} (1 + it_j) c_{qjr} \right|^2 = |c_{uhw}|^2 \prod_{jqr \in P} (t_j^2 + 1). \qquad (18)$$

(Note: in the above, $t_0 = z_0 = 1$, corresponding to the fixed orientation of the anchored link.)

Step 4. One now has several options. If the mechanism has k degrees of freedom, we may seek a k-parameter description of the solution space of the mechanism by simply regarding k of the t_j (for example) as parameters rather than unknowns. (This is the "do nothing" option.) Another idea is to add (before we start) artificial extra hyperedges whose k shape parameters will serve as the k parameters.

[11] A "hypertree" [87] is a connected hypergraph without cycles. A "cycle" in a hypergraph is a cyclic sequence of hyperedges and vertices alternately, containing at least two different hyperedges, and with no vertex or hyperedge occuring more than once, and such that each vertex is contained in both hyperedges sequentially adjacent to it in the cycle.

This can simplify the hypergraph structure tremendously in some cases.

Step 5. Finally, eliminate the unknowns! We cannot delve into the theory of elimination and resultants [30] here, which is what is generally required. However, we will point out that any particular equation of the form (EQ 18) may be solved for any particular t_j in it (in terms of all the others) by solving a univariate quadratic[12]. This observation goes a long way on the comparatively simple mechanisms of the most practical interest – sometimes enabling their complete solution. But there are mechanisms (a minimal example is shown in figure 9) for which it alone will not suffice.

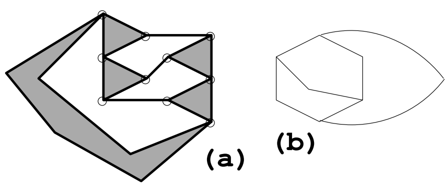

Figure 9: The simplest 1-degree of freedom mechanism (a) whose graph (b) has girth 5. ($V = 8$, $E = 10$.) For mechanisms with girth ≥ 5, the method of this section will be unable to solve for all the unknowns t_j by merely solving successive univariate quadratic equations.

Step 6. We've shown how to parametrically express the coordinates of the rivets (and hence of any point of any link of) an anchored mechanism. But what if we wish to anchor a different link? Will we now have to redo all the calculations? No! Let the new anchored link be L, whereas the previously anchored link was A. Then the coordinates $z = z_R - 0$ of some rivet R, given that link A was anchored at the origin 0 oriented so that $z_A \equiv 1$, transform (if we now regard L as the anchored link, anchored at $0'$ and oriented so that $z'_L \equiv 1$) to become

$$z' = (z_R - 0')/z_L, \tag{19}$$

where primes indicate transformed coordinates. Since we already had parametric expressions for z_L and z_R, we now get one for z'. Thus, we can still get parametric expressions for the coordinates of everything, even after a change of anchorage.

Example 7 (Yet another expression for the 3-bar curve) *If this method is applied to the 3-bar linkage of figure 5, we get*

$$|a + it_1 a - 2 - 2i - c - it_2 c|^2 = (1 + t_1^2)(1 + t_2^2) \cdot 2b^2. \tag{20}$$

Hence if one of $\{t_1, t_2\}$ is known, the other may be deduced by solving a quadratic equation. Then,

$$z = \frac{1 + it_1}{t_1 + i}(1 - f) + \frac{1 + it_2}{t_2 + i} f. \tag{21}$$

[12]Of course the roots of $Az^2 + Bz + C = 0$ are $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{2C}{-B \mp \sqrt{B^2 - 4AC}}$.

This gives a simple parameterized description of the 3-bar curve $z(t)$ and note f is now actually allowed to be general *complex* constant, i.e. the rod of length b in figure 5 is now allowed to be a triangle. Here z is located at $z = (1 - f)z_1 + f z_2$ where z_1 and z_2 are the left and right corners (both rivets) of the triangle, and z is the 3rd corner (the "pen").

5 The 6-bar (5-bar if anchored) mechanisms

To illustrate the power of the general method of §4, we now present the first full analysis of the curves traced by points on the 5-bar anchored mechanisms with 1 degree of freedom.

There are only two combinatorial types of generic 6-bar non-anchored 2D mechanisms with 1 degree of freedom, illustrated in figure 10.

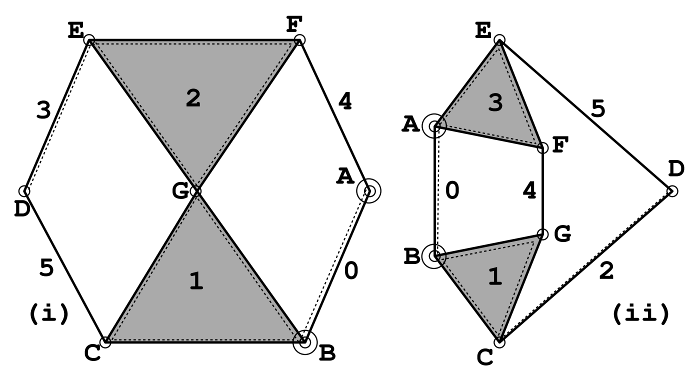

Figure 10: The two types of 6-bar 2D mechanisms: (i) "double-quad;" (ii) "quad-pent." Hypertrees used in the analysis are shown dashed. All links are numbered and all rivets are labeled.

For the purpose of analysing the double-quad, it is boring to consider anchoring one of the triangular links, since then any point on any link will trace either a circle or a 3-bar curve, and we will learn nothing new. Hence, we will only consider anchoring a rod, WLOG the one shown, numbered '0.'

We could argue similarly that for the quad-pent, WLOG we should[13] anchor rod $\overline{CD}$. However, this choice of anchor, while fully general, leads to more complicated algebra because the quadratic-solving trick of step 5 becomes insufficient. So it is a better idea to anchor $\overline{AB}$ and then regain full generality by later using the anchor change transformation trick of step 6.

So in both cases, we will assume WLOG that the two anchorpoints are at (complex number) coordinates $z_A = 0$ and $z_B = 1$.

[13]If were to anchor one of the triangles or one of the two leftmost rods, then all points on all links would trace circles or 3-bar curves, except for points on the two rightmost rods (indeed only one of them, if a triangle were anchored) which trace new kinds of curves. Hence, we may as well demand that one of the two rightmost rods (WLOG $\overline{CD}$) be anchored, since then the triangles and their connecting rods will trace the same new kinds of curves.

With these choices (the ones labeled '0', that is) for the anchored link, we choose the hypertrees shown dashed in figure 10 for "step 1" of our general analysis method.

5.1 *Equations for the double-quad*

Then the equations for the double-quad (EQ 18) arising respectively from the non-hypertree links 4 and 5 are

$$\left|(1+i)+(1+it_1)c_{B1C}+(1+it_2)c_{G2F}\right|^2 = (t_1^2+1)(t_2^2+1)\cdot 2|c_{F4A}|^2, \tag{22}$$

$$\left|(1+it_3)c_{D3E}+(1+it_2)c_{E2G}+(1+it_1)c_{G1C}\right|^2 = (t_1^2+1)(t_2^2+1)(t_3^2+1)\cdot |c_{C5D}|^2. \tag{23}$$

(We have used the facts that $t_0 = 1$ and $c_{A0B} = 1$.)

Now we may solve (EQ 22) for t_2 by solving a quadratic, and substitute the resulting expression into (EQ 23). But then (EQ 23) will also be a quadratic which may be solved for t_3. The result is that the unknowns t_2 and t_3 have been expressed in terms of the single parameter t_1. The coordinates of the rivets are

$$z_A = 0, \quad z_B = 1, \tag{24}$$

$$z_G = 1 + \frac{1+it_1}{t_1+i}c_{B1G}, \tag{25}$$

$$z_E = z_G + \frac{1+it_2}{t_2+i}c_{G2E}, \tag{26}$$

$$z_D = z_E + \frac{1+it_3}{t_3+i}c_{E3D}, \tag{27}$$

$$z_C = 1 + \frac{1+it_1}{t_1+i}c_{B1C}, \tag{28}$$

$$z_F = z_G + \frac{1+it_2}{t_2+i}c_{G2F}. \tag{29}$$

The coordinates of any point on any link are now readily expressed in terms of the t_i, e.g. a point on link 3 has coordinates $(1-f)z_E + f z_D$ where f is a (generally complex, if link 3 is allowed to be polygonal) constant depending on the point.

And so we have a 1-parameter expression (the parameter is t_1) for the double-quad curve[14].

5.2 *Equations for the quad-pent*

The equations for the quad-pent (EQ 18) arising respectively from the non-hypertree links 4 and 5 are

$$\left|(1+i)+(1+it_1)c_{B1G}+(1+it_3)c_{F3A}\right|^2 = (t_1^2+1)(t_3^2+1)\cdot 2|c_{G4F}|^2, \tag{30}$$

$$\left|(1+i)+(1+it_1)c_{B1C}+(1+it_2)c_{C2D}+(1+it_3)c_{E3A}\right|^2 = (t_1^2+1)(t_2^2+1)(t_3^2+1)\cdot 2|c_{D5E}|^2, \tag{31}$$

We may now solve the quadratic (EQ 30) for t_3 and substitute the result into (EQ 31). Then we may solve the quadratic (EQ 31) for t_2, thus expressing both t_2 and t_3 in terms of the single parameter t_1.

We may now again write expressions for all the rivet coordinates, and then apply the trick of (EQ 19) to allow any other link, in particular link 2, to be the anchored one instead of link 0.

5.3 *Line drawing devices and conicographs*

A "conicograph" is a mechanism that draws a conic curve.

Hart [35] [37] found line-drawing 2D mechanisms with only 5 polygonal links (figure 12), which is minimum possible[15].

COMPLETE CHARACTERIZATION???

Hart also found 7-link 2D mechanisms capable of drawing any conic section (quadratic curve). He conjectured that 7 was minimum possible.

How can we try to resolve this conjecture? More generally, how can we try to find special parameter choices which cause a mechanism to trace a curve of particularly low degree? One method is as follows. Suppose we have found a parametric expression $z(t) = x(t) + iy(t)$ for the curve. Then we may apply one of the following tests.

Observation 8 *A smooth and smoothly parameterized curve $z(t) = x(t) + iy(t)$ is a* line segment *if and only if*

$$\det\begin{pmatrix} \dot x(t_1) & \dot y(t_1) \\ \dot x(t_2) & \dot y(t_2) \end{pmatrix} = 0 \tag{32}$$

for all t_1, t_2 in some interval. A condition equivalent to this but involving only one t is

$$\det\begin{pmatrix} \dot x(t_1) & \dot y(t_1) \\ \ddot x(t_1) & \ddot y(t_1) \end{pmatrix} = 0, \quad \text{i.e. } \det\begin{pmatrix} \dot x & \dot y \\ \ddot x & \ddot y \end{pmatrix} = 0, \tag{33}$$

and more essentially equivalent conditions are:

$$\mathrm{Im}\big(\overline{\dot z(t_1)}\dot z(t_2)\big) = 0, \ \ \mathrm{Im}\big(\overline{\dot z}\ddot z\big) = 0, \tag{34}$$

$$\mathrm{Im}\big(\dot z/\ddot z\big) = 0, \ \ \mathrm{Im}\big(\ddot z/\dot z\big) = 0. \tag{35}$$

Observation 9 *A smooth and smoothly parameterized curve $z(t)$ is a* circle of radius R *if and only if*

$$\frac{i}{R}\dot z|\dot z| = \ddot z \tag{36}$$

where R is regarded as positive if the circle is being traced counterclockwise, otherwise negative. I.e. it is a circle (of any radius, assuming it is nonzero) if and only if

$$\mathrm{Re}\frac{\ddot z}{|\dot z|\,\dot z} \ = \ \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\ddot z}{\dot z|\dot z|}\right) \ = \ 0. \tag{37}$$

[14] I have not written out this expression in fully expanded form because it would be many pages long, but if the text and equations of this section are viewed as a multiformula algorithmic recipe rather than just one formula, then it is not at all complicated.

[15] Proof sketch: From §8.2.1 we see that the only 2D mechanisms with 1 degree of freedom are 1-link, 3-link, and 5-link devices (aside from degenerate 4 and 2-link ones which really have fewer links). But clearly the 1 link compass draws circles not lines, and the 3-bar curves of §3 cannot be lines either. Hence 5 is best possible.

There is a similar condition obeyed by general conic curves.

Observation 10 *A smooth and smoothly parameterized curve $z(t) = x(t) + iy(t)$ is a conic (i.e. quadratic algebraic curve) if and only if*

$$\det \begin{pmatrix} x_0 & y_0 & x_0y_0 & x_0^2 & y_0^2 \\ x_1 & y_1 & x_1y_1 & x_1^2 & y_1^2 \\ x_2 & y_2 & x_2y_2 & x_2^2 & y_2^2 \\ x_3 & y_3 & x_3y_3 & x_3^2 & y_3^2 \\ x_4 & y_4 & x_4y_4 & x_4^2 & y_4^2 \end{pmatrix} = 0 \qquad (38)$$

for all t_0, t_1, t_2, t_3, t_4 in some interval. In the above, x_k and y_k abbreviate $x(t_k)$ and $y(t_k)$. Also, if instead x_k and y_k abbreviate the kth t-derivative of $x(t)$ and $y(t)$ respectively, the statement remains valid.

Remarks. (i) It is possible to produce a condition equivalent to (EQ 38) but expressed purely in terms of complex numbers z_k, by simply writing $x_k = z_k + \overline{z_k} = z_k + |z_k|^2/z_k$ (and so on), then expanding the determinant by brute force. I do not know whether any especially *simple* formula may be obtained like this.

(ii) Also of course similar determinantal conditions for a smooth and smoothly parameterized curve to be algebraic of degree $\leq D$ may be written (each row of the matrix is simply all nonconstant monomials of degree$\leq D$).

(iii) Therefore an algorithm to determine whether there are any 5-bar 2D mechanisms capable of drawing general conic sections – thereby either refuting Hart's conjecture by finding a 5-bar conicograph (and 5 would be minimum possible), or proving Hart's conjecture (because 7 would be minimum possible, if there are no 5-bar conicographs) – is as follows. Compute a parametric expression for the 5-bar curves (we already have) and solve for the coefficients (defining the mechanism's sizes and shapes) to make that expression satisfy (EQ 38).

Unfortunately when I tried to run this algorithm, the final expression (EQ 38) was too large for my computer algebra system to handle, although all expressions obtained prior to that point, could be found in a minute or so of compute time. So a complete solve seemed infeasible.

But since algorithmically (EQ 38) is easy to evaluate at any particular numerical values for the coefficients, it still seemed feasible either to guess lucky satisfying values for the coefficients (which, if exact arithmetic were used, would still provide a rigorous existence proof), or to try to apply a numerical approximate equation solver to find satisfying values.

6 Mechanism menagerie

6.1 Kempe's mechanisms

6.2 Hart's mechanisms

6.3 Micromanipulator

A mechanism to compute

$$z_1 + .1(z_2 + .1(z_3 + .1(z_4 + \ldots + .1z_n))) \qquad (39)$$

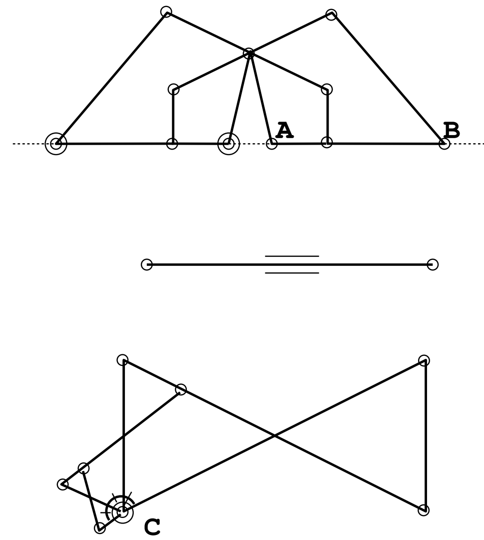

Figure 11: Mechanisms due to A.B.Kempe. The "drawbridge" mechanism at top implements a "slider" (middle); The line segment AB can move, but only along the dashed line within a bounded interval. At bottom, we show Kempe's angle triplicator. The three angles at anchored point C remain equal. (Can continue the construction to get a mechanism with $2N + 2$ bars to divide an angle into N equal parts; here $N = 3$.) The angle N-sector works because all three crossed parallelograms are similar since they each have an angle in common with another, and the side ratios are the same. The drawbridge works for the same reason; the two small "kite" quadrilaterals are similar to the two large ones, causing their angles to be the same and forcing the lines to be collinear.

could serve as a "micromanipulator." Adjusting the jth "control" z_j at length scale 1 would serve to adjust the output's coordinate at length scale 10^{-j}. Various micromanipulators have been built [27] based on hydraulic, piezolectric, and screw mechanisms, mainly. The present proposal seems more elegant mathematically than any of these, although it may not be as good from an engineering perspective.

6.4 Negative Poisson ratio

Figure 13 shows how a planar mechanism could underlie a material with negative Poisson ratio. Such materials have been made [55] [16] [17] by compressing foams with cell sizes $\approx$ 2mm. It was also discovered recently [96] that single crystals of the natural crystalline form "α-cristobalite" of quartz (SiO_2) have a Poisson ratio ranging from from $-.50$ to $+.08$, depending on the direction, with polycrystalline aggregates having Poisson ratio $-.16 \pm .03$. Apparently this is due to the SiO_2 tetrahedrons acting in a manner quite similar to a jointed mechanism.

According to [55] there are fundamental reasons why the Poisson ratio of any d-dimensional material must lie in $[-1, 1/(d-1)]$. Almost all natural materials have positive Poisson ratio, i.e. if they are compressed in one direction they expand in the traverse directions. (Incompressible ma-

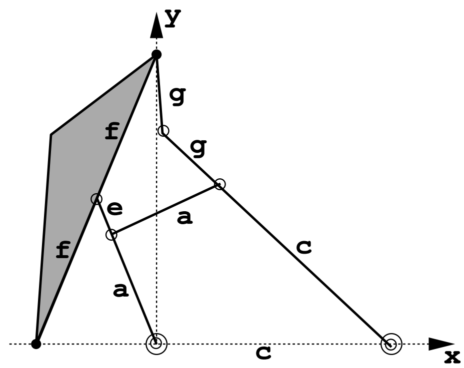

Figure 12: A 5-bar crossing-free line drawing mechanism due to H.Hart. If $f = a+e = c^3/(d^2-a^2)$, $g = a^2c/(d^2-a^2)$, $e = a^3/(d^2-a^2)$, the two dark points move along the x and y axes, drawing perpendicular lines. Any other point on the link containing these two points, e.g. the third triangle vertex, traces ellipses.

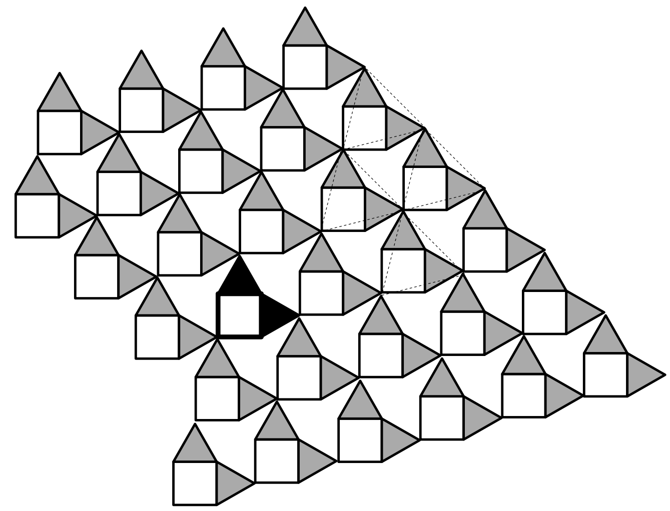

Figure 13: A "crystal" mechanism with Poisson ratio -1. (All vertices are assumed to be jointed.) The "unit cell" (shown dark) is a skew pantograph whose 3 vertices always form an equilateral triangle despite flexing, although the size of that triangle varies. By linking these triangles together to form the equilateral triangle lattice (some of which is shown dashed at right) we get an "isotropic" medium which, if it is compressed in one direction, will compress by the same ratio in every direction. Note the Poisson ratio remains constant and isotropic throughout a nonzero strain range.

terials achieve the upper bound.) Materials with Poisson ratio -1, the most negative value possible, are predicted to have some remarkable properties [55] but have never been found – but as the present geometrical construction (almost certainly the simplest possible) shows, are certainly possible in principle. A 3D material with Poisson ratio -1 may be constructed by making all the minimal norm equilateral triangle sublattices of the FCC point lattice be[16] the 2D construction in figure 13.

Although other mechanisms [1] have been proposed with negative Poisson ratio, none are as simple as the one in figure 13.

7 Hall of records

Table of best known planar mechanisms for implemented various functionalities. OMITTED???

8 Enumerating the combinatorial types of generic 2D mechanisms

8.1 Graph theoretic characterization

Theorem 11 *There is a 1-1 correspondence between* {*the combinatorial types of the (generic) 2D mechanisms with V links which (generically) have precisely $k \geq 0$ degrees of freedom if one of the links is held fixed*} *and* {*The graphs G with V vertices and E edges where $2E = 3V - 3 - k$ and such that any subgraph S induced by a strict subset of the vertices*[17] *must have $2E_S \leq 3V_S - 4$ if $V_S \geq 3$, where V_S and E_S are the respective numbers of vertices and induced edges in S.*}

Proof. Explanation of the correspondence: The vertices in the graph correspond to the rigid objects in the mechanism; the edges correspond to the hinged joints between the objects. ("Generically," a joint links exactly two objects, not three – since a 3-object joint could be regarded as two 2-object joints which just happen to be coincident – and since an edge links two vertices, that is right.) $2E = 3V - 3 - k$ because: $3V$ is the number of degrees of freedom of the V objects ($2V$ positional degrees of freedom and V rotational degrees), $2E$ is the number of constraining equations, and if we view one of the objects as being held in place, removing 3 degrees of freedom, we want k to remain (generically), explaining[18] the "$-3-k$." The subgraph conditions are because we want all subassemblies to be (generically) nonrigid; it is senseless to count a rigid subassembly as "more than one" object. If $2E_S \leq 3V_S - 4$ there won't be enough constraining equations to cause rigidity, since there will be fewer equations than variables. At least, this would be true in algebraically complete fields such as the complex numbers. In the real field, while it is possible to have rigidity even with fewer[19] equations than unknowns, this is not possible *generically*. □

[16] The FCC lattice is the integer linear combinations of the 12 lattice points closest to $(0,0,0)$, i.e. the permutations and sign changes of $(0,\pm1,\pm1)$. Each lattice point has four 2D equilateral triangle sublattices through it, for example the one generated by $(0,1,1)$ and $(1,1,0)$.

[17] I.e., including all edges ij where i,j are in the vertex subset.

[18] The formula $3V - 2E - 3$ for the number of degrees of freedom of a 2D mechanism (when applicable) is often called "Grübler's formula," although it had been stated by Chebyshev 30 years before Grübler.

[19] For example, consider $x_1^2 + x_2^2 + \ldots + x_n^2 = 0$. In atypical examples such as this, the intersection of the complex solution hypersurface with the reals is a single point. Generically, hypersurfaces are locally linear.

Remarks. (i) "Mechanisms" with 0 degrees of freedom are of interest as "primitive rigid frameworks," where by "primitive," we mean such that no subassembly is a rigid framework. Mechanisms with 1 degree of freedom are of interest as "curve tracers" and as computors of functions mapping the reals to reals. Mechanisms with 2 degrees of freedom are interesting as computors of functions of 2 real arguments or one complex argument. And so on.

(ii) All such graphs corresponding to mechanisms with k, $k \in \{0,1,2\}$, degrees of freedom are automatically 2-edge-connected[20]. To prove this, assume the contrary. In that case we would either have an unconnected graph, or a graph with a cut edge. Unconnected graphs correspond to uncoupled mechanisms – which clearly have at least 3 degrees of freedom. Some connected component S would necessarily have $2E_S > 3V_S - 4$, ruling out this possibility. Meanwhile graphs with a cut edge correspond to mechanisms with two subassemblies coupled only via a single joint – a situation of no interest to us, since if each subassembly were flexible (as we assumed) the combination would again have at least 3 degrees of freedom. Again one of the 2-connected components would necessarily[21] have $2E_S > 3V_S - 4$, ruling this out.

Mechanisms corresponding to V-vertex graphs which have $k \ge V$ degrees of freedom, are automatically disconnected; the most degrees of freedom a connected V-vertex graph can have, is $V-1$, achieved by trees. The most degrees of freedom a 2-edge connected V-vertex graph can have, is $V-3$, achieved by V-cycles.

(iii) All such graphs with $k \ge 0$ are automatically triangle-free, because triangles are rigid (this is a special case of the subassembly generic nonrigidity conditions) *except* that we allow the triangle itself as a 0-degree of freedom primitive framework.

(iv) Consideration of nongenericity would seem to be much harder[22]. If one is interested in counting 3D generic mechanisms with hinged joints, the same theorem holds but with the conditions "$2E = 3V - 3 - k$, $2E_S \le 3V_S - 4$" replaced by "$5E = 6V - 6 - k$, $5E_S \le 6V_S - 7$." However, nongenericity becomes more interesting in higher dimensions because, crudely speaking, there are more, and more interesting, ways to be nongeneric.

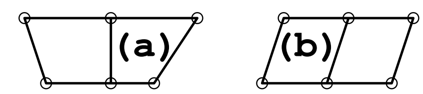

Figure 14: This mechanism is generically rigid (a). But when both quadrilaterals are parallelograms (b), we get a nongeneric flexible special case.

(v) There are no such graphs (or mechanisms) with $k \le 2$ degrees of freedom when $V \le 3$, because no 2-edge-connected triangle-free graph with $V \le 3$ vertices exists.

(vi) There are no such graphs (or mechanisms) if $V \equiv k$ mod 2, since $2E = 3V - 3 - k$ mod 2 is $k \equiv V - 1$ mod 2.

(vii) *Rooted* V-vertex graphs (that is, graphs with a distinguished "root" vertex) of our type correspond to $(V-1)$-link *anchored* mechanisms, where the Vth link corresponding to the distinguished vertex, is the "wall."

(viii) When $V = 4$ there is a unique graph with $k = 1$ degres of freedom – the 4-cycle – and it leads to a unique anchored mechanism, the 3-bar linkage (§3).

(ix) When $V = 6$ there are two such graphs, leading to two 6-link mechanisms and to five anchored 5-link mechanisms, see figure 15.

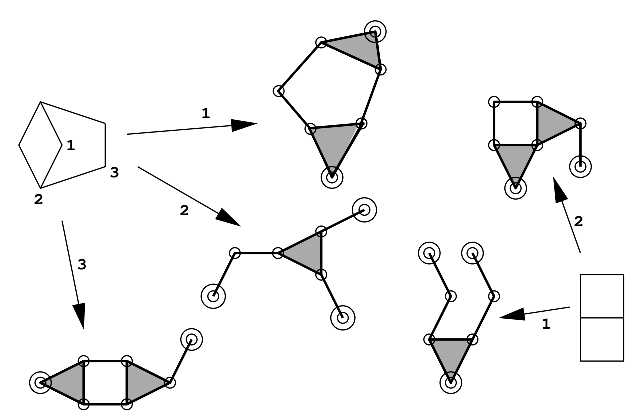

Figure 15: The two combinatorial types of generic 6-link planar mechanisms with 1 generic degree of freedom. We show the two graphs, and the three and two respective 5-link anchored mechanisms that result from the three (respectively two) different ways to root each graph at a vertex. (The link corresponding to the root vertex becomes the "wall.")

(x) Although we have regarded two generic mechanisms as "distinct" if their graphs differ, there remains the disturbing possibility that there might exist two generic mechanisms with different graphs, but which (e.g., for a 1 degree

[20] A graph is "c-edge-connected" if one must remove at least c edges in order to disconnect the graph.

[21] Consider the two components that would result if a cut edge were removed. Let component i have V_i vertices and E_i edges. Then $V_1 + V_2 = V$ and $2(E_1 + E_2) = 2(E-1) = 3V - 5 - k$. If $k \in \{1,2\}$, this certainly forces either $2E_1 \ge 3V_1 - 3$ or $2E_2 \ge 3V_2 - 3$ (or both) to be true.

[22] Generically, with N variables constrained by $N-1$ equations, we expect 1 degree of freedom. But if the coefficients in those equations are special, it is entirely possible to have either fewer or more degrees of freedom. Example: the 4-gon plane mechanism is generically nonrigid, but is rigid if the sum of three of the rod lengths is equal to the fourth rod length. The mechanism of figure 14 is nonrigid, although a naive counting of variables and constraining equations would suggest otherwise. Other examples: The "poristic polygons" ([24] and references therein) are more flexible than they should be. Kempe [52] investigated all flexible cases of a certain class of generically rigid 7-link mechanisms. The 4-gon *3D* mechanism with hinged joints is generically rigid, but is nonrigid if (1) the hinge axes happen to be parallel lines (so that the "3D" mechanism is really 2D), (2) have collinearities, or (3) form Bennett's "skew isogram" [5] (which might better be called a "twisted parallelogram," since alternate sides are equal length) in which the 4 sides of the quadrilateral form a Hamiltonian circuit of edges of a tetrahedron possessing an axis of symmetry, and in which the hinge lines are perpendicular to the tetrahedron faces defined by the two edges. ??GIVE COORDS?? Another famous 3D example: convex triangulated polyhedral frameworks (with ball joints on the rod ends) are always rigid [73], but in special cases, nonconvex combinatorially equivalent versions of them are flexible [11] [4] [20]. A remarkable recent result [22] is that all such flexes must preserve volume.

of freedom mechanism) generate exactly the same set of algebraic curves, if their length parameters are appropriately chosen. It is not known whether this "evil twin" algebraic equivalence phenomenon ever happens. Obviously if it does, then an infinite set of further derived evil twins (incorporating the original case as a submechanism) will arise.

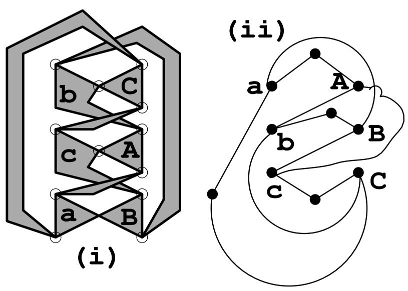

Figure 16: The simplest mechanism (i) with a nonplanar graph (ii). The graph is $K_{3,3}$ with 4 extra vertices (the unlabeled ones) put in. ($V=10$, $E=13$.) Of course, an equivalent mechanism could be made with only 1 crossing, e.g. by making the links resemble starfish made of slightly thickened half-edges (i.e. all the half-edges adjacent to a vertex) of the graph.

8.2 Counts

Let $f_k(V)$ denote the number of V-link generic planar mechanism combinatorial types[23] with k degrees of nonrigidity.

8.2.1 Tables of counts by Henry Cejtin

B.McKay [63] discovered methods for isomorph-free generation of combinatorial objects. Similar methods were discovered independently by Henry Cejtin. Both used them to count various kinds of graphs. At my request, Cejtin specialized a version of his program to enumerate 2D mechanisms. Here's a rough explanation of how Cejtin's program works in our particular case. $(V+1)$-vertex graphs are generated inductively from V-vertex graphs. Throughout, only "canonical" forms of graphs, e.g. those whose adjacency matrices are lexically minimal under all permutations of the vertices, are kept; non-canonical graphs are discarded. This eliminates redundant isomorphs. It may be implemented to run surprisingly quickly by viewing the generation processes (both of graphs and of permutations) as trees whose non-minimal branches are pruned as soon as they are detected. For an *average* graph, it is extremely easy to prune almost everything away; although especially symmetric graphs could exist which cause the canonicity tester to take a long time, this apparently does not matter much because such graphs are rare. Also, Cejtin maintains a list of all pairs (a,b) of vertices whose transposition is a graph automorphism, and his permutation generator does not generate permutations equivalent via such transpositions to a lexically smaller permutation. Graphs disobeying any subgraph flexibility condition are also discarded.

[23] That is, isomorphism equivalence classes.

Finally, the graphs were bucketed according to how many vertices and degrees of freedom they had, and what their edge-connectivity was, and the counts in each bucket were put in the auxiliary table.

8.2.2 Asymptotic behavior

Theorem and Conjecture 12 *With k fixed as $V\to\infty$,*

$$\left(\frac{3V}{4e}\right)^{V/2}V^{-O(k)} < f_k(V) < \left(\frac{e^5 V}{27}\right)^{V/2}V^{O(k)}. \qquad (40)$$

Here the lower bound is rigorous, but the upper bound depends on an unproved assumption (although rigor may be recovered by multiplying it by $V!$).

Proof. A lower bound on the asymptotic behavior of $f_k(V)$ may be derived by explicit construction. Recall that V must be even. As a graph, we use a random 3-regular graph with V vertices, but with $1+k$ edges (chosen at random) removed, for a total of $3V/2-1-k$ edges. If this graph is not 2-edge-connected, we reject it and try again. Assuming the nodes are labeled, the number (before rejections and edge removals) of possible such graphs is [62] [95]

$$\frac{e^{-2}(3V)!}{288^{V/2}(3V/2)!}\left(1+O\left(\frac{1}{V}\right)\right) \qquad (41)$$

One may easily see that the probability of rejection goes to 0 as $V\to\infty$ (e.g. see chapter VII section 6 of [13]). An induced subgraph with V_S vertices has all valencies ≤ 3 and hence has $3V_S/2$ edges at most. Unfortunately, this is up to 2 more edges than our desired upper bound permits. However, if the graph is 2-edge-connected – which it is, since otherwise we rejected it – that means at least two edges of our graph must join our subset to its complement, and these edges are not available to be part of the $3V_S/2$. Hence, we have the desired upper bound $3V_S/2-2$. Dividing by $V!$ because we are interested in counting unlabeled graphs (note: it is known [95] that the number of unlabeled r-regular V-vertex graphs, $r\ge 3$, is indeed asymptotic to $1/V!$ times the number of labeled ones when $V\to\infty$), we conclude that

$$f_k(V) > \frac{(3V)!}{288^{V/2}\,V!\,(3V/2)!}V^{-O(k)} = \left(\frac{3V}{4e}\right)^{V/2}V^{-O(k)}. \qquad (42)$$

An upper bound on the asymptotic behavior of $f_k(V)$ as $V\to\infty$ with $k\ge 1$ fixed, may be derived as follows. First, the number of vertex labeled V-vertex graphs with E edges is $\binom{(V-1)V/2}{E}$, i.e., since $2E=3V-3-k$ in our case, $\binom{(V-1)V/2}{(3V-3-k)/2}$. In fact, we need only consider the subset of such graphs which happen to be 2-edge-connected and triangle free, which presumably would tighten the bound by a factor of c^V at least, for some $c>1$. But we won't bother. If we crudely estimate that asymptotically, almost all of these graphs (at least, if they are 2-edge-connected) have no automorphisms, the effect of *not* labeling the vertices is essentially the same as dividing by $V!$, and then by the crudest version $x! = (x/e)^x x^{O(1)}$ of Stirling's formula we hypothesize that the number of V-vertex graphs with E edges where $2E = 3V - 3 - k$ is approximately

$$(\frac{e^5 V}{27})^{V/2} V^{O(k)}. \tag{43}$$

This is asymptotically an upper bound on $f_k(V)$. Of course we never justified our "crude estimate," but rigor may be regained by multiplying by $V!$, and presumably, a factor much smaller than $V!$ still suffices. □

9 Kempe, Planar mechanisms, and algebraic completeness

The fact that planar mechanisms can draw any algebraic curve, has been incorrectly attributed by a number of sources [3] [88] to A.B.Kempe[24]. Actually, Kempe never showed this, although he probably could have done so; some rather inefficient proofs were given by later authors [53] [28] in the period 1877-1902.

Recently a great deal of excitement [65] [66] [86] [75] [76] [77] has been generated concerning geometric "algebraic universality theorems." However, the fact that the first such theorem dates in essence to the 1870s, remained unnoticed. Our purpose in this section is to re-demonstrate that theorem and re-view it from a modern perspective.

We will give an elegant proof of a more general result and now even allowing true planarity to be demanded:

Theorem 13 (Algebraic completeness)
(1) Let $P(x, y)$ be any polynomial in two variables. There exists an anchored 2D mechanism which constrains a point on it to lie on the curve $P(x, y) = 0$; but the point is free to move along ths curve within some finite range of motion, which may be made as large as desired.

(2) Also, there exists a mechanism with n points on it whose coordinates (now regarded as complex numbers) are z_1, z_2,..., z_n such that the mechanism enforces $P_j(z_1, z_2, ..., z_n) = 0$ for $j = 1, ..., m$ $(1 \leq m \leq n)$ where the P_j are any specified multivariate polynomials.

(3) Also, there exists a mechanism constraining n points on it to lie on n given line segments, but free to move anywhere within them, and enforcing $P_j(x_1, x_2, ..., x_n) = 0$ for $j = 1, ..., m$ $(1 \leq m \leq n)$ where the P_j are any desired multivariate polynomials and where x_1, x_2,..., x_n are the distances along the line segments.

(4) In 1,2,3 above, we may also demand any desired finite set of polynomial inequalities $P_i(x_1, x_2, ..., x_n) \geq 0$. In all cases, if the polynomial inequalities and equalities have no solution, then no mechanism will exist, otherwise it will exist.

(5) In 1-4 above, we may also demand that the given polynomial equalities and inequalities are the only ones that apply within some bounded region of coordinate space (which may be made as large as desired), i.e. the mechanism supports no other solutions.

(6) 1-5 still hold even if we also demand the mechanism be true planar.

(7) Conversely, the full range of motion of any finite 2D mechanism is described by a finite set of polynomial equalities and inequalities.

Proof. Claim (7) is immediate from the theory of resultants as in §4. To demonstrate (2), it will suffice to show

1. How to duplicate and transmit information (in the form of bounded complex displacements) from anywhere in the plane, to anywhere else;

2. How to perform the four fundamental binary operations $z_1 \times z_2$, z_1/z_2, $z_1 + z_2$, $z_1 - z_2$ (as well as rescalings cz and translations $c \pm z$, where c is a constant, which we may view as unary operations);

3. How to impose inequality and equality constraints $x_j \leq x_k$, $x_j \leq c$, etc.

The remaining claims will follow from (2) by adding a few additional bells and whistles. The details follow.

Linear arithmetic operations.

Scaling by any fixed *real* ratio may be accomplished with the aid of the pantograph, first described in print by Christoph Scheiner[25] in 1630.

The 4-link "skew pantigraph" of Sylvester[26] [89] (figure 18) computes the general "affine combination" of two complex variables

$$z_2 = (1 - c)z_0 + cz_1 \tag{44}$$

[24] Alfred Bray Kempe (1849-1922) was a barrister and amateur mathematician. He graduated from Cambridge in 1872 and was elected to the Royal Society in 1881. He is best known for his ill-fated "proof" of the 4-color theorem, announced in Nature 21 (July 17, 1879) p. 399 and published in the American Journal of Mathematics 2 (1879) 193-201. Kempe retracted the proof after the error was pointed out by P.Heawood 10 years later. An obituary of Kempe is in Proc. Roy. Soc. London A 102 (1923), i - x.

[25] Christoph Scheiner was born in Wald, near Mindelheim in Swabia (southwest Germany) in 1573; he died in 1650 in Neisse Silesia, now Nysa Poland. Scheiner attended the Jesuit Latin school in Augsburg, continued at the Jesuit college at Landsberg, and entered the Jesuit order in 1595. He then entered the university at Ingolstadt in 1600 to study metaphysics and mathematics. In 1610 he joined the faculty as professor of Mathematics and Hebrew. Scheiner is best known as the early astronomer who wrote the main book [82] on sunspots. Scheiner (re)discovered sunspots in 1611 and showed that the sunspot rotation axis was 7.3° off the plane of the ecliptic. Scheiner was the inventor or an early constructor of a "camera obscura" and helioscope devices, and an early user of telescopes; and he also showed that the retina was the seat of vision. Supposedly the pantograph was invented by an artist who kept its principle of operation a secret; Scheiner then reconstructed and unveiled it [83]. Scheiner used a pantograph in his observations of sunspots [82]. A massive but little known refutation of the Copernican theory of planetary motion ("Introductory Treatise in Favor of a Moving Sun and a Stable Earth against Galileo Galilei") by Scheiner was published postumously in 1650. Scheiner was also the loser in a dispute with Galileo concerning the nature of sunspots. (Scheiner initially subscribed to the incorrect and religiously inspired view that the sun must be perfect and without blemishes, but eventually the data convinced him of the opposite.)

[26] J.J. Sylvester (1814-1897), a famous British mathematician, perhaps best known for his invention of the determinant formula for "resultants." Sylvester's lecture [90] on Peaucellier's inversor was the cause of all the other work by British researchers on linkages. Obituaries: Nature 39 (1889), 217-219; Mathematische Annalen 50 (1898) 133-156.

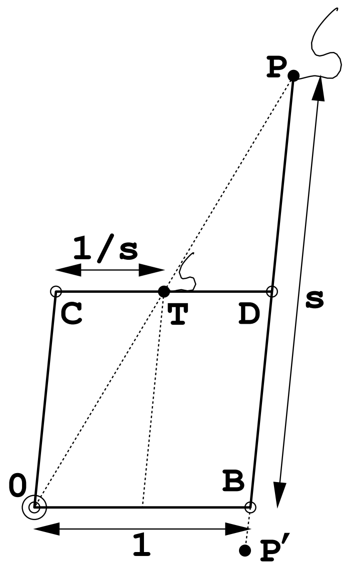

Figure 17: Pantograph. $0CDB$ is a rhombus of side 1, and $0CTA$ is a parallelogram. Because $\triangle 0AT$ and $\triangle 0BP$ remain similar during flexure, pen P traces scaled version of curve at tracer T. Scaling factor is s, i.e. if T has coordinates (x, y), then $T = (sx, sy)$. $s = 2$ is shown, but $s < 0$ is also possible, e.g. replacing P by P' would yield $s = -.2$. Range of motion: P can move anywhere within the annulus $s - 1 \leq \sqrt{x^2 + y^2} \leq s + 1$.

where c is any complex constant ($c = se^{i\psi}$ in the notation of figure 18) and z_0 and z_1 are arbitrary complex number inputs within the range of motion $s-1 \leq |z_1 - z_0| \leq s+1$ of the device. In fact any 2-variable linear function $z_2 = az_0 + bz_1$, where a and b are complex constants, is computable by using two such devices. Note that this allows complex addition, rotation, translation, and scaling.

Adding a complex constant ($z_2 = z_1 + c$) may instead be accomplished with a simpler 6-bar doubly anchored mechanism, the 2-parallelogram "translator" of figure 19.

Duplication and transmission of information.

The rotation, scaling, and translation devices above will trivially transport complex number information from anywhere to anywhere else. However, if we demand true planarity – no two bars are allowed to cross – then how are we supposed to allow information transit paths to cross? The simple solution is the crossover device of figure 20.

Nonlinear arithmetic operations.

In 1875, H.Hart[27] [35] discovered the crossed parallelogram "inversor" of figure 21.

Previous less efficient inversors[28] had been discovered by N.Peaucellier and L.Lipkin[29].

Notice that, since the graph of the inversor (i.e. a 4-cycle)

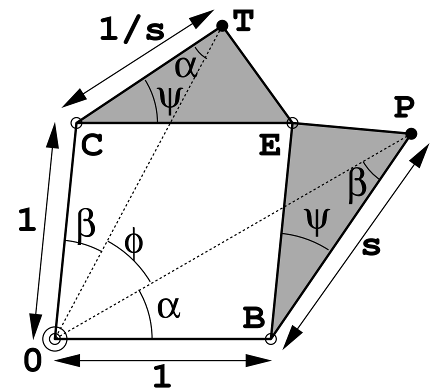

Figure 18: Sylvester's generalization, the "skew pantigraph." $0BEC$ is a rhombus and $\triangle BET$ is similar to, but scaled s times larger in linear dimensions than $\triangle CPE$. By side-side-angle, $\triangle 0BT$ and $\triangle PC0$ remain similar (but the the former is scaled s times larger) during flexure. Since $\angle E + \angle B = 180°$ and $\alpha + \beta + \psi + \angle B = 180°$, we see that $\alpha + \beta + \psi = \angle E$. But $\alpha + \phi + \beta = \angle E$. Therefore $\phi = \psi$ remains constant during flexure. The scaling by s and the constant angular rotation by ψ implies that the (complex number) coordinates z_P of the pen P are $z_P = z_0 + (z_T - z_0)se^{\subset\psi}$.

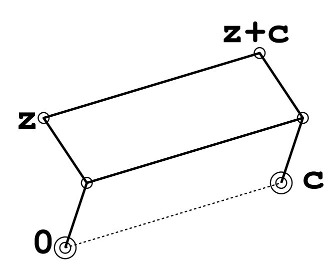

Figure 19: 2-parallelogram translator performs $z \to z + c_2 - c_1$. Range of motion: $|b - a| \leq |z| \leq b + a$.

[27]Harry Hart (1848-1920) was born in Greenwich, Kent, son of hotel keeper Charles Hart. He obtained a B.A. (1871; "fourth wrangler") and M.A. (1874) from Trinity College Cambridge, also becoming a "fellow" in 1873. He became a mathematics instructor at the Royal Military Academy in Woolwich, and published a number of papers on geometry. He is not to be confused with Andrew S. Hart (1811-1890) a geometer at Trinity College of Dublin. Thanks to Val Tucci & John Smith for unearthing this information from Cambridge alumni records.

[28]Since inversors map circles to circles, they are useful for drawing circular arcs of too large radii to be drawn with a compass. Also, a circular arc tangent to the center of inversion, is inverted to a line segment. This allows the exact conversion of rectilinear reciprocating motion (such as the piston of a steam engine) to circular reciprocating motion (the "rocking beam" of a steam engine) purely by means of linkwork. Thus Peaucellier, Lipkin, and Hart's inversors, and Kempe's "drawbridge" mechanism (figure 11), solved exactly – after 80-90 years – the straight line motion problem posed, and approximately solved, by James Watt (1736-1819), developer of the steam engine. This explains the prizes awarded by the French and Russian governments.

[29]Charles Nicolas Peaucellier (1832-1913) was a French "Capitaine de génie à Nice." He announced his "inversor" linkage in 1864 – in the form of a question and without explaining the solution – in a letter to the Nouvelles Annales. The 14th edition of the Encylopedia Brittanica has information on him. Eventually Peaucellier became a general and (Sylvester claimed) was in command of the fortress of Toul. After a rediscovery by a Russian student, L.I.Lipkin (1851-1875) in 1871 (Lipkin died before completing his doctoral dissertation) and his demonstration of a working model at the World Exhibition in Vienna 1873, Peaucellier published the details [71]. Sylvester claims the French government awarded Peaucellier the "Prix Montyon" (1875) for his invention, whereas Lipkin received a "substantial reward from the Russian government."

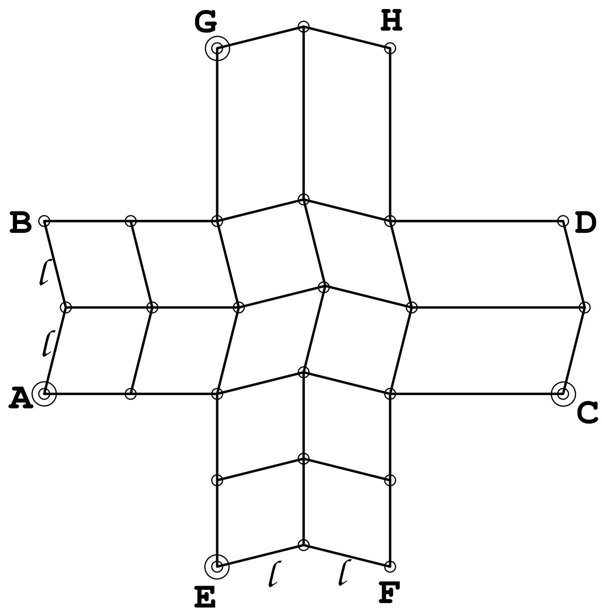

Figure 20: Information crossover device. All 16 quadrilaterals shown are parallelograms (any sizes and shapes will work). The parallelograms force, e.g. the 2-bar polygonal line connecting G to H, to be a pure translate of the 2-bar polygonal line EF. Hence the (complex number) coordinates of the labeled vertices obey $z_B - z_A = z_D - z_C$ and $z_H - z_G = z_F - z_E$. Hence if we anchor z_A, z_C, z_E, and z_G, then complex number information will be transmitted from B to D and from F to H. If the four "arms" are made sufficiently long and skinny and the 4 lengths labeled "ℓ" are made of length ℓ, then the range of motion is $|z_B - z_A| \le 2\ell$ $|z_F - z_E| \le 2\ell$.

is planar, it actually isn't necessary for any bars to cross; see figure 22.

Then Kempe and Sylvester both realized that the same "skewing" transformation that Sylvester had applied to the pantograph to get his skew pantograph, could also be applied to Hart's crossed-parallelogram inversor to get the skew inversor of figure 23. Of course this also need not involve a crossing.

If 0 is anchored at the origin, the skew inversor transforms $z \to m/\overline{z}$, but more generally by allowing 0 to have the complex coordinates z_0, the skew inversor will compute

$$z_2 = z_0 + m/\overline{z_1 - z_0} \tag{45}$$

where m is any complex constant, and z_0 and z_1 are arbitrary complex number inputs within the range of motion $|\overline{Bz} - \overline{0B}| \le |z_1 - z_0| \le \overline{Bz} + \overline{0B}$ of the device. The overline in (EQ 45) denotes complex conjugation. (Notice that the upper [and lower] bounds may be made as large [and small] as desired.) In particular, this includes all nonlinear Moebius (also called "fractional linear") transformations of $\overline{z_1}$.

All algebraic operations.

If multiplication is available, then division is available also by running the multiplier "backwards" (or vice versa).

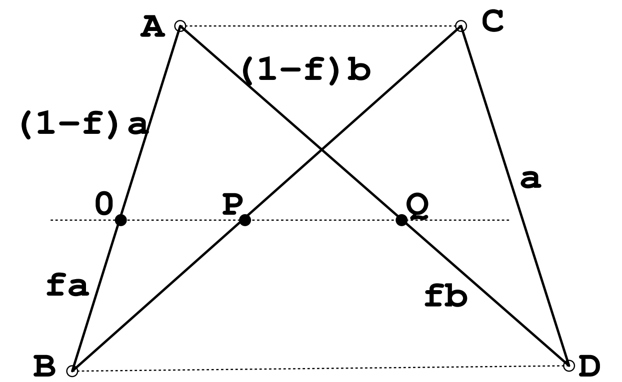

Figure 21: Hart's crossed parallelogram "inversor." Three points O, P, and Q remain collinear as the mechanism flexes, and the product of the distances $\overline{0P}$ and $\overline{0Q}$ remains constant. Proof: The collinearity of O, P, and Q, and the facts that $\overline{0P} = (1-f)\overline{BD}/2$ and $\overline{0Q} = (1+f)\overline{AC}/2$, are by similar triangles (e.g. $\triangle ABD \sim \triangle AOQ$). Hence $\overline{0P} \cdot \overline{0Q} = (1-f^2)\overline{BD} \cdot \overline{AC}/4$. Now we use "Ptolemy's theorem" (a hoary special case of Bowman's equation (EQ 11)) that the product of the lengths of the diagonals (here b^2) of an inscribable quadrilateral $ABDC$ is equal to the sum of the products of pairs of opposite sides (here $\overline{BD} \cdot \overline{AC} + a^2$) to see that $\overline{BD} \cdot \overline{AC} = b^2 - a^2$. Hence $\overline{0P} \cdot \overline{0Q} = (1-f^2)(b^2-a^2)/4$.

If scaling by a factor of 4, addition, and subtraction are available, then 2-operand multiplication is not needed; merely 1-operand squaring ($z \to z^2$) will suffice, since $4ab = (a+b)^2 - (a-b)^2$. (And square rooting is accomplishable by running the squarer backwards.) Even squaring is not needed[30] since we have an inversor ($z \to c/\overline{z}$), because

$$z^2/c = c\Big/\overline{\overline{c}/\overline{z} - \overline{c}/\overline{z+c}} - z. \tag{46}$$

Inequality constraints; real arithmetic. So far, we have shown that any set of algebraic equalities among n *complex* variables may be imposed by a 2D mechanism – indeed, even a true-planar one, free of crossings – within some

[30] See [54], exercise 4.6.4.37 to further appreciate the the power of transformations of the form (EQ 45).

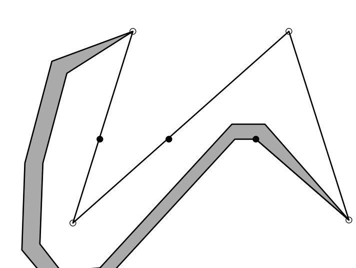

Figure 22: Hart's crossed parallelogram inversor can be built without any crossings, and with all three key points accessible from the "outside." Dashed lines show example of way to restrict the range of motion to prevent the parallelogram from "uncrossing," by adding 2 extra bars.

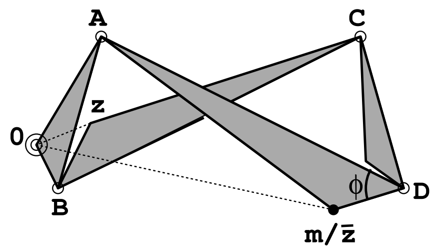

Figure 23: 4-bar "skew inversor." The 4 grey triangles are similar and $\overline{AB} = \overline{CD}$, $\overline{AC} = \overline{BD}$. The "modulus" of the inversor is $m = (\overline{AC}^2 - \overline{AB}^2)(1-\mu)\mu$ where the general complex quantity μ describes the shape of the grey triangles, i.e. $z = \mu C + (1-\mu)B$, i.e. $\mu = e^{i\phi}\overline{Bz}/\overline{BC}$.

bounded range of motion (which may be made as large as desired by sizing the parts).

If we instead desire the n variables to be *real*, this is easily achieved by using n Hart straight line motion devices (figure 12) to constrain the variables to intervals on the real axis (or suitable translations of it). By choosing the bounds on these real intervals, we may impose any inequalities $\ell_k \le x_k \le u_k$ we want (i.e. any algebraic inequalities whatever) and if we size the parts correctly, all the other undesired inequalities arising from the bounded ranges of motion of each component mechanism may be rendered irrelevant (provided no real quantity or intermediate quantity ever becomes too large).

Solution uniqueness.

The mechanisms we have described could allow other motions (other branches of algebraic curves), besides the algebraic relationships we desired. If one wishes to forbid that, this is easily accomplished with the methods of §3.2 for preventing crossing, uncrossing, and folding of parallelograms (and notice these methods can still be done in a crossing-free way as in figure 22).

GO THRU CLAIMS 1-7!!

□

9.1 Some remarks on 3D mechanisms

10 Degree bounds

REDO!!?? BETTER RESULTS MAY BE OBTAINED BY MAKING VERY STRAIGHT APPROXIMATE LINE MECHANISMS BY RECURSING ON THE CHEBYSHEV-TYPE CONSTRUCTION.

Theorem 14 *A pen mounted on a bar of a 2D anchored mechanism with B bars and one degree of freedom will trace an algebraic curve whose degree is $\le 2^{B+2}$. On the other hand, for every odd $B \ge 1$ there exists a 2D (even pure planar) B-bar mechanism of this kind which traces a curve of degree $\ge 2^{B-6}$.*

It is easy to see that curves of degree c^B may be produced. By starting with a real variable x constrained to a line segment (by means of a Hart 5-bar line-drawer) and repeatedly squaring it n times, we may compute x^{2^n} with a mechanism with $B = 5+Kn$ bars, where K is the minimum number of bars are required to build a squarer (including suitable provisions for transmission of input and output).

For a better bound, one may repeatedly iterate the function $f(x) = \frac{1}{2}(x+x^{-1})$, since[31] $f^{[n]}(x)$ is a rational function whose numerator has degree 2^n, and since $f(x)$ may be computed by an anchored mechanism with only 6 bars (figure 24). We thus reach degree 2^n using $5+6n$ bars.

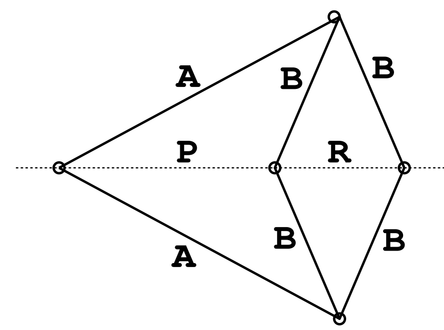

Figure 24: Peaucellier's mechanism. $R = (B^2 - A^2)/P - P$.

But the best lower bound I have – and it is obtained with a pure planar mechanism – is gotten by concatenating anchored 3-bar linkages (figure 25). Each linkage performs the transformation $P_{n+1} = f(P_n)$ where P_n is the *squared* diagonal length of quadrilateral n. $f(P)$ may be deduced by finding the other squared diagonal length with the aid of Bowman's equation (EQ 11) and then using the law of cosines to deduce the angle subtended by it, and hence by the next quadrilateral's diagonal, and hence P_{n+1}. With the particular choice $a = 4$, $b = 3$, $c = 5$, $d = 6$, which has the advantage that all the vertical quadrilateral sides may be made precisely vertical,

$$f(P) = \frac{89P - P^2 - 77 + (P-1)\sqrt{(P-49)(P-121)}}{5P}. \tag{47}$$

If the final output is combined with the input via a 2-bar linkage, the result (i.e. the middle joint of the 2-bar linkage) will trace a curve of degree $\ge 4^n$ and the mechanism requires only $2n+3$ bars, $n \ge 2$.

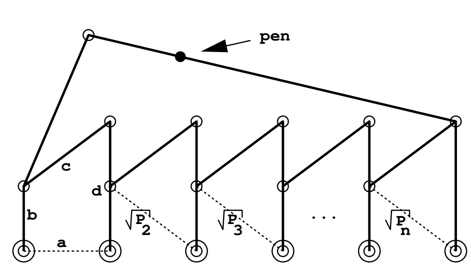

Figure 25: $2n+3$ bar mechanism for tracing a curve of exponentially high degree.

(We also need to deal with the initial cases 1, where a compass draws a circle, of degree 2; 3, where we get a 3-bar

[31] $f^{[n]}(x)$ is the nth iterate of Heron's algorithm for approximating $\sqrt{1}$ with initial guess x.

curve, of degree 6; and $B = 5$???)

The first statement is also easy – if we wheel in known degree bounds for multivariate multipolynomial resultants. If we simply write down the obvious quadratic equations obeyed by the inter-rivet distances and apply proposition 1.1 chapter 13 page 427 of [30], we conclude that the algebraic curve described by a rivet has degree $\leq 2^B$ and the algebraic curve described by a point anywhere on some link has degree $\leq 2^{B+2}$. □

11 NP (and otherwise) complete problems; undecidability

The main results of this section are theorems 15, 16, 17, 18 and 19.

Theorem 15 *Determining whether the global minimum value of a quartic multivariate polynomial with integer coefficients lies below given some threshold K (or even determining the minimum value to within an additive amount ± 1, or to within a multiplicative constant factor $1+\delta$, for some sufficiently small absolute constant $\delta > 0$) is strongly*[32] *NP-complete.*

Theorem 16 *Determining whether the origin is a local minimum of a multivariate quartic polynomial is strongly NP-complete.*

Theorem 17 *Given a finite set of multivariate polynomial inequalities and two points (as rational coordinate tuples). The "algebraic connectedness problem" (of determining whether there exists a continuous path connecting the two points, every point of which obeys the inequalities) is PSPACE-complete.*

Theorem 18 *Given a finite set of multivariate polynomial inequalities. The "real algebraic satisfiability problem" (of determining whether there exists a set of real values for the variables causing all the inequalities to be satisfied) is NP-hard.*

The problem in theorem 18 has been called the "existential theory of the reals (ETR)."

Theorem 19 *Given a finite set of multivariate inequalities, each side of which is expressible as a formula involving using integer constants, the function $\sin(\pi x)$ and its inverse function $\arcsin(y)/\pi$, parentheses, and the four arithmetic operations $+$, $-$, $\times$, $/$. The "trigonometric connectedness problem" (of determining whether there exists a continuous path connecting the two points, every point of which obeys the inequalities) is Turing-undecidable.*

Proof sketch for theorem 15: The idea is first, as a lemma, to show that finding (or approximating) the global min of an n-variate quadratic on the 2^n vertices of a hypercube, is NP-complete. Then, we devise a quartic with 2^n equal local minima, located at the 2^n vertices of a hypercube. Finally, we show that an appropriate linear combination of the quartic and the quadratic will have its global min very close to a hypercube vertex and that the min of the quadratic will be deducible from the min of the quartic.

To explain the steps in more detail:

Lemma 20 *It is NP-complete to find the minimum value of a quadratic function of d variables, each lying in the set $\{0,1\}$. Indeed, we construct an infinite set of quadratics with bounded integer coefficients, so that even approximating the minimum value to within a factor of 1.01, or to within an additive constant, is NP-hard.*

Proof. Wlog, the quadratic only has terms of the forms $x_i x_j$, $(1-x_i)x_j$ $x_i(1-x_j)$, and $(1-x_i)(1-x_j)$, $1 \leq i < j \leq n$, with coefficients in $\{0,1\}$, There are no linear terms (x_i). Also wlog the hypercube we are speaking of is $\{0,1\}^n$. In that case, maximizing the quadratic is exactly the same problem as entirely general MAX-2SAT: maximizing the number of clauses $x_i \wedge x_j$, $\bar{x}_i \wedge x_j$, etc. from a list of order S specified clauses, which are satisfied by n boolean variables x_1, ..., x_n. MAX-2SAT is known to be MAX-SNP-hard and indeed it is known that approximating the maximum achievable number of clauses to within 1% is NP-hard [2]. □

Now to prove minimizing quartics is hard, we first note that $p(x) = (x^2-1)^2$ has exactly 2 local minima at $x = \pm 1$, with value 0. Hence

$$\text{quartic}(x) = p(x_1) + p(x_2) + \ldots + p(x_d) \tag{48}$$

has 2^d equal local minima at the vertices "$(\pm 1)^d$" of a d-hypercube. Now, add ϵquadratic(x) to this quartic where ϵ is very small. The heights of the minima will be perturbed by $\epsilon\text{quadratic}(x) + O(\epsilon^2)$ and their locations will be perturbed by $-\frac{\epsilon}{8}\vec{\nabla}\text{quadratic}(x) + O(\epsilon^2)$. It will now suffice to take $\epsilon = 0.001/S$ where S is the sum of the absolute values of the (bounded integer) coefficients of quadratic(x), and then rescaling all the coefficients to integers by multiplying by $1000S$ still leaves the mean square coefficient bounded by an absolute constant, if $S = \Omega(d^2)$. □

Proof of theorem 16: Let us be in $(n+1)$-dimensional real space and regard the vectors of this space as 2-tuples $(\vec{x}, z)$ where $\vec{x} \in \mathbf{R}^n$. Let $Q(\vec{x})$ be the n-variate quartic from theorem 15. Then

$$z^4 Q(\frac{\vec{x}}{z}) \tag{49}$$

is an $(n+1)$-variate quartic such that the origin $(\vec{x}, z) = (\vec{0}, 0)$ is a local minimum if and only if the global minimum of $Q(\vec{x})$ is positive. □

Proof of theorem 17: There are two parts to this proof: (a) we need to prove that solving the problem is PSPACE-hard, and (b) we need to show that it is possible to solve the problem in PSPACE.

Part (a) is accomplished as follows. We observe that one can build a computer with N bits of memory (e.g., a Turing machine with N "tape squares") out of $O(N)$ rigid

[32] That is, still NP-complete even if the coefficients of the polynomial are required to be written in unary.

mechanical 3D parts, such that it is possible to disentangle one of the parts from the others so that it can escape to infinity, if and only if that computer "halts." The parts are described by a finite set of polynomial inequalities and the conditions that say no two parts can intersect is described also by a finite set (indeed of cardinality $O(N^2)$) polynomial inequalities. This variant of the halting problem is (obviously) PSPACE-complete[33]

To clarify our "observation" above: the fundamental shapes of engineering, such as haflspaces, ellipsoids, cylinders, and cones, are all describable as polynomial inequalities in 3 variables. Now, suppose we want to take the intersection, union, or complement of two fundamental shapes, to get a more complicated shape. (For example, convex polyhedra are the intersections of halfspaces; nonconvex polyhedra are the unions of convex polyhedra.) This may be accomplished as follows. Let two shapes be described by $\vec{F}(\vec{x}) \geq \vec{0}$ and $\vec{G}(\vec{x}) \geq \vec{0}$. Then their intersection is

$$\sum_j [F_j(\vec{x}) - z_j^2]^2 + \sum_k [G_k(\vec{x}) - y_k^2]^2 = 0 \tag{50}$$

where the z_j and y_k are extra variables. We have used the trick that the inequality $a \geq b$ is writeable as the equality $a - b - c^2 = 0$, and then use the trick that $p = 0$ and $q = 0$ if and only if $p^2 = q^2 = 0$. The union of the two shapes is

$$\left(\sum_j [F_j(\vec{x}) - z_j^2] \right) \left(\sum_k [G_k(\vec{x}) - y_k^2] \right) = 0 \tag{51}$$

since $p = 0$ or $q = 0$ if and only if $pq = 0$. The complement of the interior of the first shape is

$$\vec{F}(\vec{x}) \leq \vec{0}. \tag{52}$$

Clearly by continuing in this manner we may express any Boolean combination of fundamental engineering shapes as a set of polynomial inequalities. This should make it clear (at least to the modern reader) that all useful engineering shapes may be expressed this way, and that a computer may be constructed.

Part (b) arises from a known result of J.Canny [14]. □

Proof of theorem 18: This result is well known. One simple proof is to put equalities $(1 - x_j)x_j = 0$ forcing every variable x_j to be in the set $\{0, 1\}$, and then, given any boolean feedforward circuit, by replacing $a \,\mathrm{AND}\, b$ by ab, $a \,\mathrm{OR}\, b$ by $a + b - ab$, NOT b by $1 - b$, we get a polynomial, which we set equal to 1. Then the polynomial equalities can be satisfied if and only if the boolean circuit was "satisfiable" – and this is the well known NP-complete "SAT" problem. □

Proof of theorem 19: The proof proceeds in three steps. The first step is to observe that it is undecidable whether iteration, starting from a given natural number X, of a certain piecewise linear function $f(x)$ mapping the natural numbers to the natural numbers, will ever reach 0. The second step is to observe that we can build an "integer detector" using the fact that $\sin(\pi x) = 0$ if and only if x is an integer. The third step is to build a set of inequalities which force transitions from integers to integers to be precisely the ones of the form $x \leftarrow f(x)$. We'll now explain these steps in more detail.

Step 1: The magic function $f(x)$ is

$$f(x) = f_{x \bmod 6}(x) \tag{53}$$

where f_0, f_1,..., f_5 are certain explicit linear functions of x with the property that f_j maps nonnegative integers which are j mod 6, to nonnegative integers. Explicit choices of the 12 coefficients in the f_j may be deduced from Minsky [64]'s treatment[34] demonstrating the undecidability of whether X, $f(X)$, $f(f(X))$,..., ever reaches 0. This function may be rewritten as the integer-argument special case of a continuous function mapping reals to reals:

$$f(x) = 6 \sum_{j=0}^{5} \frac{\sin(\pi x)}{\sin([x - j]\pi/6)} f_j(x). \tag{54}$$

Steps 2,3: As the proof of theorem 17 above made clear, we can write any Boolean AND/OR/NOT combination of inequalities as a set of inequalities. Let us now use this fact. Consider two reals v and w, regarded as positions of points on two parallel lines in 3-space To be specific, let the 3-space have Cartesian coordinates (x, y, z); let these two lines be $\{x = \pm 1, y = 0\}$; and the two points are $(1, 0, v)$ and $(-1, 0, w)$. Let "condition C" be the statement that v and w are both integers (i.e. $\sin(\pi v) = \sin(\pi w) = 0$) and $w = f(v)$. If condition C is satisfied, then we allow a 3-space rotation to take place about the line $x = y = 0$. (This "allowing" may be done by adding suitable inequalities to our set, i.e. the two points will be (x, y, v) and $(-x, -y, w)$ with $x^2 + y^2 = 0$.) If this rotation is 180°, then v and w are interchanged. If condition C is not satisfied, the continuous rotation is disallowed, and only the two extreme rotations (that is, at 0° and 180° so that our two points both have $y = 0$) are allowed. If our two points both do have $y = 0$, then we demand that v be an nonnegative integer, but allow w to be an unrestricted real.

We are now done: transitions between extreme-rotation states with v a nonnegative integer can occur only if $v_{\mathrm{new}} = f(v_{\mathrm{old}})$ or the reverse. Either way, the extreme-rotation state with $v = 0$ is reachable from an initial such state with $v = X$ if and only if $f(f(f(...X))) = 0$. □

11.1 Applications of these theorems to planar mechanisms

Our algebraic completeness theorem 13 shows that, by making our planar mechanisms move in a horizontal plane

[33] An interesting special case of this was addressed by [41], who showed a motion planning problem for rectangles sliding inside a rectangle, was PSPACE-hard.

[34] That treatment first shows the Turing computational universality of "two-counter machines." Then, by regarding the number of 3's and 2's in the prime factorization of x as the values of the two "counters," the result follows. This idea was essentially due to J.H.Conway, who invented a computational model called "FRACTRAN," although Conway thought three counters were required, whereas Minsky realized that two could be made to suffice.

("tabletop") and attaching some point on that mechanism to a string connected through a hole in the table to a weight, we can make a mechanical system whose potential energy function is any desired polynomial function.

Theorem 15 then shows that it is NP-complete to find the minimum potential energy state of this system. Theorem 16 then shows it is NP-hard even to tell if a totally specified configuration of the mechanism is in a local potential minimum.

Theorem 17 shows that the question of whether one particular specified configuration of a planar mechanism, is reachable from another, is PSPACE-complete. Actually, even just asking if a specified point on the mechanism may be moved from within one specified disc to within another, or even if it may be moved at least a specified distance away, are PSPACE-hard.

Theorem 13 of course shows directly that the question of whether a planar mechanism whose parts are specified and whose combinatorial description is specified, is actually realizable (i.e. the rivets may be assigned coordinates in the plane that will achieve the specifications) is ETR-complete. Theorem 18 shows this question is NP-hard.

Finally, theorem 19 shows that if we extend our notion of "planar mechanism" to also allow rolling contact of lines along circular "wheels," so that we may compute $\sin(x)$, then (at least, if we are allowed to use infinitely long rays as rigid mechanical parts, and are thus no longer required to make all our parts have finite size) the two-configuration mutual reachability problem is Turing-undecidable[35].

12 Devices based on the "downhill principle" can solve NP-hard problems

As we just showed, for planar mechanisms with rigid parts, with a weight attached, it is NP-hard even to tell if a totally specified configuration of the mechanism is in a local potential minimum.

But if we assume the "downhill principle" [93] that physical systems which are not in local potential minima will go downhill, then a simple physical experiment will solve a strongly NP-hard problem in (apparently) a polynomial number of steps. Just set up the mechanism in the desired configuration, attach the weight, and let go – we are not at a potential minimum if the weight moves.

We conclude that at least one of the following must be false: {The downhill principle, $P \neq$NP, and Church's strong thesis}. This conclusion seems paradoxical because of three of these are widely believed.

Vergis et al. [93] had noticed a similar result and were unable to resolve the paradox. We will show that the downhill principle is the culprit.

13 The downhill principle cannot be relied upon [in noiseless quantum physics]

Firstly, in quantum physics, there is a sense in which the downhill principle is "more than true," i.e. not only will a body tend to decrease its potential energy, but in fact it can do so even if there is no downhill path available, by "tunnelling" through a potential barrier. In theorem 21 we show that in fact tunneling is *unavoidable*. One might perhaps have thought that, perhaps by using some symmetry, some barrier could be designed that there would be no inclination to tunnel through, but as the theorem shows, this is impossible.

Secondly, and for our purposes more importantly, there is a strong sense in which the downhill principle in quantum physics *cannot* be relied upon, see theorem 22.

Neither of these theorems are difficult and they would seem to be quite basic, probably belonging in the elementary quantum mechanics books. However, they do not seem to have been stated previously[36]

Theorem 21 (Unavoidability of tunneling) *For Schrödinger's equation in the xy plane, if Ψ is 2-time differentiable everywhere, and the potential energy function $V(x,y)$ is finite everywhere, then it is impossible for $\Psi(x,y,t)$ to be 0 in some region enclosing some disc of nonzero radius for a nonzero time interval.*

Proof. Consider the Schrödinger equation

$$[V(x,y) - \frac{\hbar^2}{2m}\nabla^2]\Psi = i\hbar\dot{\Psi}. \tag{55}$$

Suppose $\Psi(x,y,t)$ were 0 in some region R (enclosing some disc of nonzero radius) for a nonzero time interval. Then $\Psi = 0$, and also $\vec{\nabla}\Psi = \vec{0}$ (by continuity of Ψ and its gradient) on ∂R. Assuming $\Psi \neq 0$ outside the boundary (and assuming the second spatial derivatives of Ψ exist everywhere), that means that it must be the case that

$$L(\vec{x}) = \frac{\mathrm{d}^2}{\mathrm{d}u^2}\Psi(\vec{x} + u\vec{n}) \tag{56}$$

is nonzero everywhere on ∂R, where $\vec{n}$ is a unit normal to ∂R. Hence $\nabla^2\Psi$ is nonzero on ∂R. Hence

$$i\hbar\dot{\Psi} = \frac{-\hbar^2}{2m}\nabla^2\Psi \neq 0 \tag{57}$$

on ∂R. Hence, instantly Ψ will become nonzero and probability density will invade the region. □

Remark. The theorem and proof above also work in any d-dimensional space ($d \geq 1$, d finite): Ψ cannot be zero everywhere in any ball, for a nonzero duration of time.

Theorem 22 (Failure of downhill principle in quantum mechanics) *There exists a everywhere smooth nowhere flat potential $V(x,y)$ with no basins anywhere[37], indeed such that there is a smooth curve starting from anywhere on which V monotonically decreases[38] to arbitrar-*

[35] I suspect that if only a finite number of *finite-size* parts described by a finite number of polynomial inequalities are allowed, then even if rolling contact along algebraic curves is allowed, the reachability problem would still be decidable, because there would only be a finite number of topologically distinct trajectory types, which would be recursively enumerable.

[36] For example, they are not mentioned in the texts by Pauli, Schiff, Messiah, or Reed and Simon.

[37] A "basin" is a connected set such that the only way to get out of it, is to go upward in potential.

[38] We may even demand this decrease have |slope| ≥ 1 everywhere.

ily negative potential energies, such that: there are "bound state" wave functions, corresponding to probability densities localized[39] *with a discrete spectrum.*

Remark. This sheds light on the often stated physicist's notion that "bound states," trapped in potential wells below the potential energy at infinity, have discrete spectra and have localized probability densities, whereas "unbound" states, with energy greater than the potentials available at infinity, have continuous spectra and have non-localized probability densities. As we now see, the situation is actually more complicated than that, since potentials which classically would be incapable of trapping anything, can indeed lead to bound states and a discrete spectrum.

In others words: "the potential goes down, but the system can't."

Proof sketch.

The potential is

$$V(x,y) = \begin{cases} x^2+y^2 & \text{if } x \le 0 \text{ or if } x>0 \text{ and } |y| \ge 2w(x) \\ -kw(x)^{-2} - u(x) & \text{if } x>0 \text{ and } y=0 \\ \text{a smooth join} & \text{if } x>0 \text{ and } |y|<2w(x) \text{ and } y \ne 0 \end{cases} \quad (58)$$

where $w(x)>0$ is a decreasing function of x which might be called the "width of the valley". Also $u(x)>0$. The potential is a harmonic oscillator parabolic well mostly, but there is a "valley" which goes downward, enabling the particle to get out of the harmonic oscillator well by flowing down the valley.

Also, k is some positive constant which has to be chosen correctly to make it work (also $u(x)$ must be chosen) but I think $k \approx 4\pi^2$ will do. More on this later.

The idea is that even though there is a downhill path to potential $-\infty$ through the valley, the valley keeps narrowing, "squeezing" any particle that tries to traverse it so much that it "prefers" not to.

After I showed Bill Bialek my proof argument (below) for this valley narrowing example, he provided the following simple intuitive explanation, based on energy conservation, of why it works. As the mass-m particle goes down the valley to reach potential $-P(x)$, say, it gets squeezed between the valley's narrowing walls of width $w(x)$. This by the $\delta x \cdot \delta p > \hbar/2$ uncertainty principle causes a kinetic "localization energy," to arise, of order $(\hbar/w)^2/m$. Hence if $w(x)^{-2}$ is of larger order than $-P(x)$, we expect it will be impossible to leak along the valley because the kinetic energy gain arising from being squeezed by the valley's walls, will arbitrarily exceed the potential energy drop caused by going further down the valley.

Bialek's argument may be made rigorous by using Lieb's alternate formulation (based on a Sobolev inequality) [58] of the $\delta x \cdot \delta p$ uncertainty principle – this is probably the best proof.

To sketch the argument I originally had in mind: we recognize that the time independent Schrödinger equation

$$\nabla^2 \Psi = \Psi V \quad (59)$$

has known solution in the large harmonic oscillator region, and then try to glue this solution onto analysis of the behavior of the solution in the remaining valley region. We do not have to understand the solution in the valley very precisely in order to show that the valley's narrowing will suffice to cut the probability current to asymptotically 0.

My claim is that if $w(x)$ and $u(x)$ are chosen so that

$$w(x) = O(x^{-2-\epsilon}) \quad (60)$$

for some $\epsilon > 0$, and

$$\sqrt{u(x)}w(x) \to 0 \text{ as } x \to \infty \quad (61)$$

and k is chosen correctly, then the wave cannot leak to infinity by flowing down the valley. Therefore, we effectively get bound states with this potential. □

14 The downhill principle cannot be relied upon [in noiseless classical physics]

Theorem 23 (Failure of downhill principle in classical mechanics I) *There exists a everywhere smooth nowhere flat potential $V(x,y,z)$ with no basins anywhere, indeed such that there is a smooth curve starting from anywhere on which V monotonically decreases*[40] *to arbitrarily negative potential energies such that: there exists a point $X = (\vec{p}, \vec{q}))$ in position-momentum phase space, such that a Hamiltonian system started from a randomly selected point within the ball of radius ϵ centered at X, will, with probability bounded by a constant above 0, trace a closed orbit that stays in that ball forever.*

Proof. The potential is

$$V(x,y) = \begin{cases} x^2+y^2+z^2 & \text{if } z<0 \text{ or if } z^2 \le (x^2+y^2)\delta_2^2 \ \& \ |y| \ge 2w(x) \\ -z & \text{if } z>0 \ \& \ z^2 \ge (x^2+y^2)\delta_1^2 \\ \text{smooth join} & \text{if } z>0 \ \& \ (x^2+y^2)\delta_1^2 < z^2 < (x^2+y^2)\delta_2^2 \end{cases} . \quad (62)$$

In other words, usually a harmonic oscillator parabolic potential well centered at $(0,0,0)$, except that in a sharply pointed cone with apex $(0,0,0)$ centered on the positive z axis the potential goes down like $-z$. Here $0 < \delta_1 < \delta_2$ are constants.

A random initial starting point chosen from a ball in phase space centered at $(0,0,0),(0,0,0)$ will orbit the origin in an elliptical orbit, if the plane of this ellipse does not intersect the cone so that all the particle ever experiences is $V = x^2+y^2+z^2$. The probability of that depends on δ_2, but anyhow is constant. □

Theorem 24 (Failure of downhill principle in classical mechanics II) *At a potential* maximum *which behaves locally quadratically, the instability is too weak to kick a particle out in finite time.*

Proof. One can exactly solve Newton's equations of motion for a point mass $m>0$ in a 1-dimensional potential of the form $U(x) = -|x|^\alpha$, $\alpha \ge 0$. Then

$$mdotx^2 = 2|x|^\alpha. \quad (63)$$

[39] That is, with a large fraction of their mass inside that ball

[40] Even with $|\text{slope}| \ge 1$, if that is desired.

The solution of (EQ 63) with $x, \dot{x} \to 0+$ when $t \to 0+$ is

$$x(t) = \left[\frac{(2-\alpha)t}{\sqrt{2m}}\right]^{2/(2-\alpha)} \quad \text{if } 0 < \alpha < 2 \tag{64}$$

Observe that $x(t)$ is an increasing function of t if $\alpha < 2$. When $\alpha = 2$, the solution is

$$x(t) = \exp\left[\sqrt{\frac{2}{m}}t\right]. \tag{65}$$

However, when $\alpha > 2$, there is no solution of (EQ 63) with $x, \dot{x} \to 0+$ when $t \to 0+$. In other words, the instability of a potential maximum of order > 2 is too weak to kick out a point mass in finite time. $\square$

Fortunately for Church's thesis in the presence of planar mechanisms made of rigid parts, as we've seen, detecting local minima is *not* NP-hard, in fact it is in P, if the putative minimum is a stationary point of order ≤ 3. Hence NP-hardness can only happen at larger orders than 3, and the downhill pronciple is never valid in such cases.

15 The downhill principle in the presence of thermal noise

MORE DETAIL???

In classical mechanics with thermal noise (brownian motion in the presence of forces) obviously the system will always be able to find a way downhill. However, if the "thermal noise" is regarded as really just an interaction with a larger external system, then by modeling the combined system we are back to the same results as before.

16 Acknowledgements

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Series analysis of Watt's [approximate line] mechanism . . . . 5-9.
On a [new approximate line] mechanism (1868) 10-15.
About parallelograms (1869) [includes "Grübler's" formula for counting degrees of freedom] . . 16-36.
On the centrifugal [speed] regulator (1871) . 37-53.
On gears . . . . 54-84.

On symmetric parallelograms (1885) [compressed description of results in earlier longer paper pp. 101-112] 85-91.
On the simplest connections (1878) . . . . 92-100.
On symmetric parallelograms (1879) . . . 101-112.
Parallelograms (1879) . . . . 113-141.
Theorem about Watt's curve (1881) . . . . . 142.
4th power approximate lines (1881) . . . 143-157.
Continuous computing machine (1882) [i.e. with rolling contacts, not toothed gears] . . . 158-160.
Converting rotational into straight motion (1884) 161-166.
Simplest linked system with motion symmetric about an axis (1888) . . . . 167-211.

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