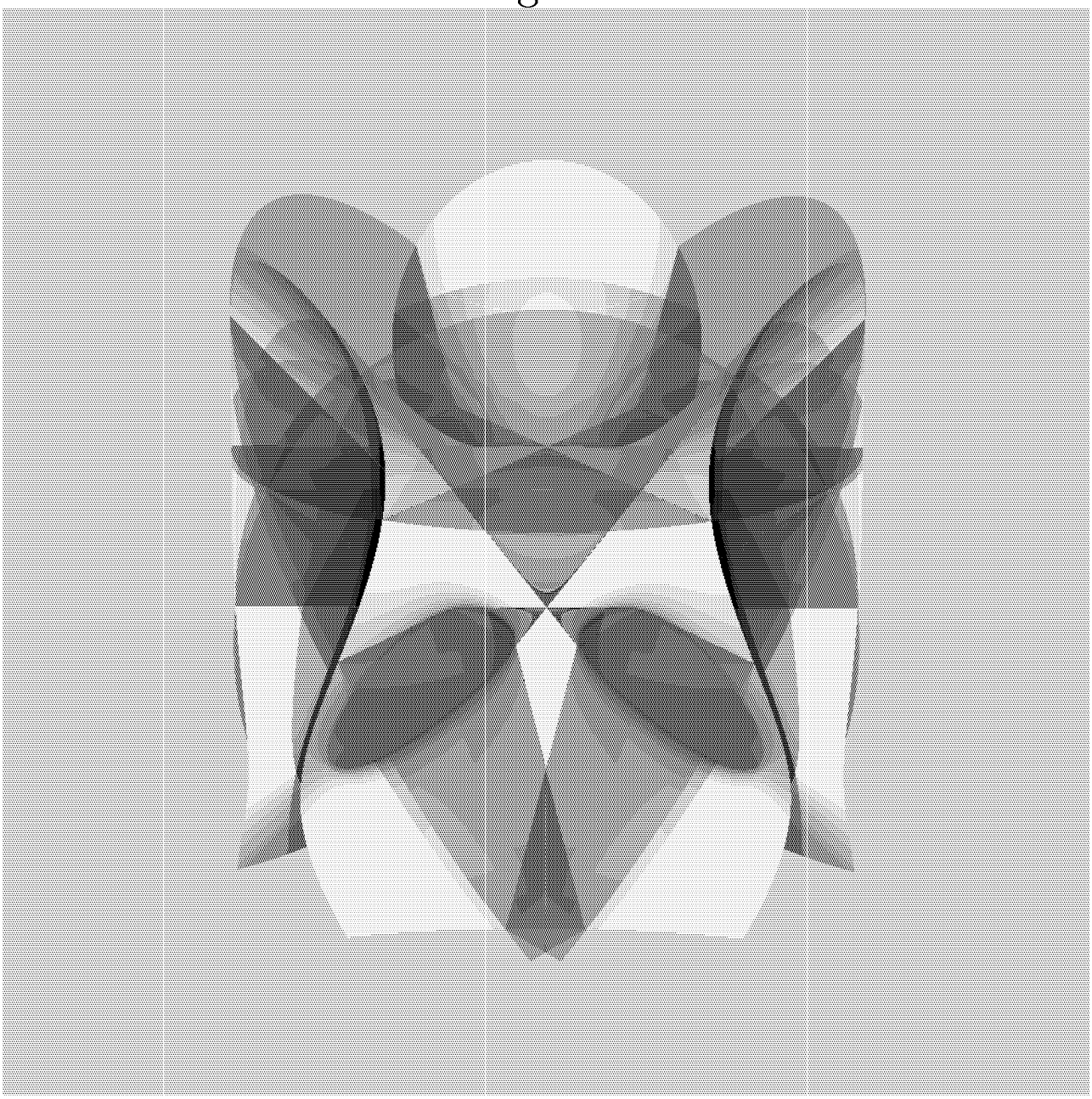
The Clebsch diagonal cubic surface



The Clebsch diagonal cubic surface is the unique surface with automorphism group equal to the symmetric group Σ_5. This can be seen from the equation, with variables $(y_1, \ldots, y_5)$ in $\mathbf{P}^4$ fulfilling $\sum y_i = 0$. The equation is then

$$y_1^5 + \cdots + y_5^3 = 0, \quad \sum_{i=1}^{5} y_i = 0.$$

The Clebsch surface is also the unique cubic surface with 10 so-called Eckard points, which are points on the surface where the three lines in a tritangent meet at a point. The picture above shows four such Eckard points (at least two of which are visible), i.e., in addition to the cubic there are 4 special tritangent planes (which meet the surface in three lines), each containing an Eckard point. The affine equation used here is

$$\begin{aligned} &81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) + \\ &\quad 54xyz + 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1 = 0, \end{aligned}$$

and the plane at infinity is given by $y_1 + y_2 + y_3 + 1/2y_4$.