

A Gem of the modular universe

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Introduction

There is a very beautiful set of modular varieties, studied in [J], which have not only nice interpretations as moduli spaces, but also have projective embeddings as very special hypersurfaces in \mathbb{P}^4 . The first is the Segre cubic \mathcal{S}_3 , the unique cubic hypersurface in \mathbb{P}^4 (threefold) with ten ordinary double points. This is the moduli space \mathbf{P}_6^1 of six marked points in \mathbb{P}^1 . The dual variety, the Igusa quartic \mathcal{I}_4 , is the moduli space of six points on a conic in \mathbb{P}^2 , birational to \mathcal{S}_3 , but with singular locus consisting of 15 singular lines which meet in 15 singular points. These two varieties also turn out to be moduli spaces of Jacobians of certain curves:

- \mathcal{S}_3 is the Satake compactification of the arithmetic quotient of the complex three-ball \mathbb{B}_3 by the principal congruence subgroup $\Gamma_{\sqrt{-3}}(\sqrt{-3})$ of $U(3, 1; \mathcal{O}_K)$, where $K = \mathbb{Q}(\sqrt{-3})$. This is the moduli space of Jacobians of trigonal (also called Picard) curves of genus 4, with a $\sqrt{-3}$ -level structure.
- \mathcal{I}_4 is the Satake compactification of the arithmetic quotient of the Siegel space of degree 2, \mathbb{S}_2 , by the principal congruence subgroup of level 2, $\Gamma(2) \subset Sp(4, \mathbb{Z})$. This is the moduli space of Jacobians of hyperelliptic curves of genus two with a marking of the six branch points.

In [J] the moduli questions were treated in detail, but the projective embeddings were only mentioned. In this paper we will provide proofs (some of which are known but difficult to find in the literature) on the projective realisations \mathcal{S}_3 and \mathcal{I}_4 of these moduli spaces. These give rise to other interesting hypersurfaces: the Hessian varieties of \mathcal{S}_3 and \mathcal{I}_4 , hypersurfaces \mathcal{N}_5 and $\text{Hess}(\mathcal{I}_4)$ of degrees 5 and 10, respectively, and another degree 10 variety \mathcal{W}_{10} , closely related to $\text{Hess}(\mathcal{I}_4)$. \mathcal{W}_{10} has the same degree, symmetry group and singular locus of $\text{Hess}(\mathcal{I}_4)$, and may actually coincide, a point we were unable to clarify. The variety \mathcal{W}_{10} is the image of \mathcal{N}_5 under the birational map of \mathbb{P}^4 into itself given by the Jacobian ideal of \mathcal{S}_3 . \mathcal{N}_5 has been studied recently by Barth and Nieto [BN].

Another beautiful modular variety is the Burkhardt quartic \mathcal{B}_4 , which can be shown to be the unique quartic hypersurface threefold with 45 ordinary double points [JSV]. The variety \mathcal{B}_4 again has a moduli interpretation:

- \mathcal{B}_4 is the Satake compactification of the arithmetic quotient of the three-ball \mathbb{B}_3 by the principal congruence subgroup of level 2, $\Gamma_{\sqrt{-3}}(2)$ in $U(3, 1; \mathcal{O}_K)$, K as above. This is the moduli space of Jacobians of trigonal (also called Picard) curves of genus 4, with a level 2 structure.

In this paper we introduce an algebraic fourfold, a quintic hypersurface \mathcal{I}_5 in \mathbb{P}^5 , which is related to all the above mentioned varieties; this is the gem of the title. But it turns out to be also related to the moduli space \mathbf{P}_6^2 of six marked points in \mathbb{P}^2 , hence also to the moduli space of marked cubic surfaces in \mathbb{P}^3 . The moduli space \mathbf{P}_6^2 has a projective realisation, which we denote by \mathcal{Y} ; it is a double cover of \mathbb{P}^4 branched along the Igusa quartic \mathcal{I}_4 , $\mathcal{Y} \rightarrow \mathbb{P}^4$. We may also consider the double cover of \mathbb{P}^4 branched along the variety \mathcal{W}_{10} , which we denote by $\mathcal{W} \rightarrow \mathbb{P}^4$. The relations to the above mentioned varieties (with the exception of \mathcal{B}_4 , whose relation to \mathcal{I}_5 is much more subtle and complex, making it impossible for us to discuss it here) are as follows.

The Segre cubic is the resolving divisor of 36 triple points of \mathcal{I}_5 , while there are 36 hyperplane sections (dual to the 36 triple points) isomorphic to \mathcal{N}_5 . Similarly, \mathcal{I}_4 and \mathcal{W}_{10} are related to the dual variety. But by far the

most intriguing is the relation to the Coble variety \mathcal{Y} , which is the main result of this paper. Let $\mathcal{Y} \rightarrow \mathbb{P}^4$ and $\mathcal{W} \rightarrow \mathbb{P}^4$ be the double covers branched over \mathcal{I}_4 and \mathcal{W}_{10} , respectively; let $\mathcal{Z} = \mathcal{W} \times_{\mathbb{P}^4} \mathcal{Y}$ be the fibre product. Considering the projection from a triple point $p \in \mathcal{I}_5$ displays \mathcal{I}_5 blown up in p as a double cover of \mathbb{P}^4 , branched along $\mathcal{S}_3 \cup \mathcal{N}_5$. Dually this is the double cover branched over $\mathcal{I}_4 \cup \mathcal{W}_{10}$. We prove

Theorem 0.1 (Corollary 9.14) \mathcal{I}_5 sits Σ_6 -equivariantly birationally in the center of the diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\quad} & \mathcal{Y} \\ \downarrow & \searrow & \downarrow \\ \mathcal{W} & \xrightarrow{\quad} & \mathbb{P}^4 \end{array}$$

This shows the relation between the quintic \mathcal{I}_5 and the Coble variety \mathcal{Y} .

So we see that \mathcal{I}_5 is *very near* to being the moduli space of marked cubic surfaces, but not quite. Let us remark here that one suspects this diagram to have the following property: all varieties $\mathcal{Z}, \mathcal{W}, \mathcal{Z}', \mathcal{Y}$ are quotients of the symmetric domain of dimension four and type **IV** (or equivalently, type **I_{2,2}**). This is true for \mathcal{Y} (see Theorem 5.9). In [M] a diagram of subgroups commensurable with the group giving \mathcal{Y} as quotient is given on p. 389. These groups are in the group $U(2, 2)$, which gives the domain of type **I_{2,2}** which by one of the exceptional isomorphisms is the same domain as type **IV₄**. The following diagram of groups:

$$\begin{array}{ccc} S\Gamma(i+1) & \hookrightarrow & S\Gamma_T(i+1) \\ \downarrow & & \downarrow \\ \Gamma_M(i+1) & \hookrightarrow & \Gamma_T(i+1) \end{array}$$

may be conjectured to correspond to the diagram of varieties above; all arrows are injective inclusions of subgroups of index two. For the variety \mathcal{Y} this is proved in [MSY]. All these groups are arithmetic subgroups $\Gamma \subset G_{\mathbb{R}}$, such that the arithmetic quotient $\Gamma \backslash \mathcal{D}$ is a moduli space of certain K3 surfaces with a marking on H^2 of some special type.

As a by-product we get the following result concerning moduli spaces of Calabi-Yau threefolds:

Theorem 0.2 (Corollary 9.9) *The family of hyperplane sections of \mathcal{I}_5 passing through one of the 36 triple points p is, via projection, a family of Calabi-Yau threefolds which are degenerations of double octics, branched along the union $V_3 \cup V_5$ of a cubic and a quintic surface which meet in 15 lines.*

In other words, the family of quintic hypersurfaces is connected with the moduli space of double octics, along a four-dimensional sublocus.

Let us now make a few remarks about the entire set of examples. First of all, they are (almost) all hypersurfaces. For a Baily-Borel embedding to be a hypersurface, the singularities on X_{Γ}^* must be hypersurface singularities — which they usually are not. Note that all of the examples where X_{Γ}^* is a normal hypersurface (i.e., excluding perhaps the dual of \mathcal{B}_4), the arithmetic groups contain torsion. Roughly speaking, this is experimental evidence of a statement like: Γ has torsion \iff the singularities of X_{Γ}^* are very mild (hypersurface, complete intersection); Γ torsion free \iff the singularities are not so mild. Whether this has some general validity would seem to be quite a difficult problem.

Secondly, all examples have interesting, decomposing hyperplane sections, and the components of these reducible sections are modular subvarieties. Let \mathcal{X} be one of our varieties, and let:

$$\begin{aligned} \mu &= \# \text{ of hyperplanes } H, \text{ such that } H \cap \mathcal{X} \text{ decomposes} \\ \nu &= \# \text{ of linear subspaces on } \mathcal{X} \text{ cut out by them} \\ \tau &= \# \text{ of linear subspaces in each } H \cap \mathcal{X} \end{aligned}$$

Then the results are summed up in Table 1.

So this behavior looks like it can be expected in general, at least for those Γ which have torsion. Some other curiosities of the examples are the following. The *Hessian variety* $\text{Hess}(V)$ of a hypersurface V meets V along the parabolic divisor, which is the locus of points where the tangent hyperplane sections has worse than an ordinary double point at the point of tangency; it gets blown down to a singular locus of the dual variety under the duality map. The following strange behavior occurs in our examples:

Table 1: Special hyperplane sections in the examples

\mathcal{X}	μ	ν	τ	rest of $H \cap \mathcal{X}$
\mathcal{S}_3	15	15	3	–
\mathcal{I}_4	10	$10 \mathbb{P}^1 \times \mathbb{P}^{1's}$	1	–
\mathcal{N}_5	15	15	3	quadric surface
	15	15	5	–
\mathcal{W}_{10}	10	$10 \mathbb{P}^1 \times \mathbb{P}^{1's}$	1	–
	?	?	?	?
\mathcal{B}_4	40	40	4	–
\mathcal{B}_4^\vee	45	$45 \mathbb{P}^1 \times \mathbb{P}^{1's}$	1	?
\mathcal{I}_5	27	45	5	–
\mathcal{I}_5^\vee	36	120	10	\mathcal{I}_4

In this table we have included in addition to linear subspaces, also quadric surfaces, which are products of linear spaces, in the column titled “ ν ”.

- i) $\mathcal{S}_3 \cap \text{Hess}(\mathcal{S}_3) = 15$ planes;
- ii) $\mathcal{I}_4 \cap \text{Hess}(\mathcal{I}_4) = \text{ten quadric surfaces}$;
- iii) $\mathcal{B}_4 \cap \text{Hess}(\mathcal{B}_4) = 40$ planes;
- iv) $\mathcal{I}_5 \cap \text{Hess}(\mathcal{I}_5) = 45 \mathbb{P}^3$'s.

Furthermore, each of these intersections consists of modular subvarieties, viewing the varieties as arithmetic quotients¹. Another classical notion is that of *the Steinerian* of a hypersurface: it is the locus of singular points of the quadric polars with respect to V , which are singular. (The Hessian is the locus of points for which the quadric polar is singular, and the vertices of these cones cut out the Steinerian.) The following is even stranger:

- i) \mathcal{B}_4 is self-Steinerian;
- ii) \mathcal{I}_5 is self-Steinerian.

It is really not understood what self-Steinerian means geometrically, but has something to do with the parabolic divisor. For i) we have a conceptual proof, due to Coble; in the latter case this has just been checked computationally.

As the expressed purpose of this paper is to point out the relation between a set of (established) modular varieties and the quintic \mathcal{I}_5 , we spend some time first recalling these varieties; their equations, moduli interpretations and birational models. Although much of this material is known to experts, we recall these for the convenience of the reader; moreover there are several facts for which we prefer to give our own proofs, making the paper in fact quite self-contained. This preparatory material is collected in Part I, while the results on \mathcal{I}_5 are given in Part II.

We end this introduction by recalling a few facts about cubic surfaces and the 27 lines on them, which is fundamental to all that follows.

Cubic surfaces

Consider a smooth cubic surface S in $\mathbb{P}^3(\mathbb{C})$, given by the vanishing of a cubic form f . There are exactly 27 lines on this surface, a fact first pointed out by George Salmon in 1849, in response to a letter from Arthur Cayley. The argument he used is reproduced for example in [SF], and is purely algebraic. For the convenience of the reader we sketch this briefly. Consider hyperplane sections H_t passing through one of the lines. Since

¹it has not yet been rigorously proved that \mathcal{I}_5 is an arithmetic quotient, as mentioned above

$H_t \cap S = \{\text{3rd degree plane curve}\}$, which already contains a line, it consists generically of a quadric and a line. Since this intersection contains two double points, it follows that *every* such plane is a bitangent, i.e., tangent to S in two points. For a finite number of hyperplanes H_t this intersection degenerates into three lines, as illustrated in Figure 1.

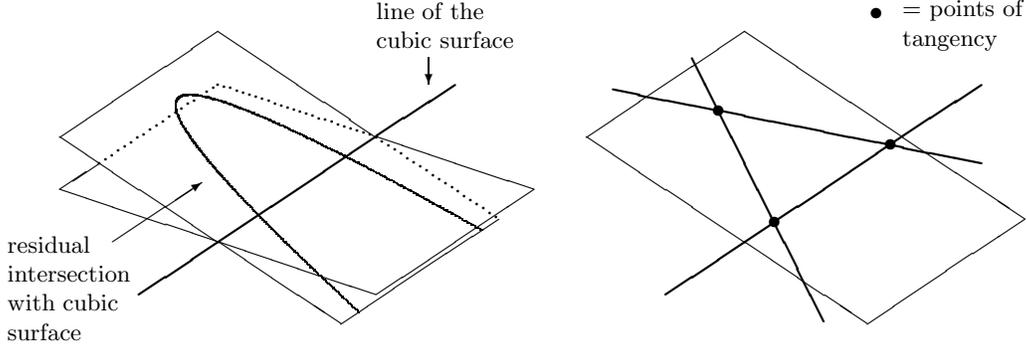


Figure 1: (a) generic section through a line (b) tritangent section

Such planes H_t which are tangent to S at three points (and contain three lines) are called accordingly *tritangent planes*. Now fix a line L on the cubic surface given in local coordinates by $x_3 = x_4 = 0$, say. Then the equation $f = 0$ can be written $x_3U + x_4V = 0$ with U, V quadratic. To find the tritangent planes one puts $x_3 = \mu x_4$ into this relation, divides by x_4 and forms the discriminant of the ensuing equation. Viewing this as an equation in μ , one sees easily that it has degree 5, i.e., there are five values of μ for which the corresponding plane is a tritangent, or in other words the given line lies in five tritangent planes. Now count: starting with a given tritangent plane, it meets $4 \cdot 3 = 12$ other tritangents, in each of which there are two other lines, which gives $12 \cdot 2 + 3 = 27$ lines on a smooth cubic surface. Since each line is contained in five tritangents, this leads to $27 \cdot 5 \div 3 = 45$ tritangent planes to the cubic surface S . In the book [SF] several other proofs of the magic number 27 are given.

We consider now the group of permutations, $\text{Aut}(\mathcal{L})$, of the 27 lines (or of the 45 planes), by which we mean the permutations of the lines preserving the intersection behavior of the lines. For this it is useful to consider the famous double sixes and the notation for the 27 lines introduced by Schläfli. A *double six* is an array

$$N = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$$

of 12 of the 27 lines with the property that two of these 12 meet if and only if they are in different rows and columns. (This notation distinguishes this particular set of 12, although any such double six is equivalent to it under $\text{Aut}(\mathcal{L})$). The other lines are given by the $\binom{6}{2} = 15$ $c_{ij} = a_i b_j \cap a_j b_i$, where $a_i b_j$ denotes the tritangent spanned by those two lines. There are 36 double sixes, namely the N above, 15 N_{ij} and 20 N_{ijk} :

$$\begin{aligned} N_{ij} &= \begin{bmatrix} a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} \\ a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} \end{bmatrix}, \\ N_{ijk} &= \begin{bmatrix} a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} \\ c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n \end{bmatrix}. \end{aligned} \tag{1}$$

Since a double six describes, by definition, the intersection behavior of the lines, we see immediately that Σ_6 (the symmetric group on six letters) acts by permutations on a double six and a \mathbb{Z}_2 acts by exchanging rows. Since there are 36 double sixes, we see $|\text{Aut}(\mathcal{L})| = |\Sigma_6| \cdot 2 \cdot 36 = 51,840$. A natural question arising here is: how many lines do two double sixes have in common? The answer is twofold:

- either: four (like a_1, a_2, b_1, b_2 , which N and N_{12} have in common) which have the property of lying in pairs in planes, the pairs being however mutually disjoint;

Table 2: Loci associated with the 27 lines on a smooth cubic surface

# objects	description of the objects
27	lines on a cubic surface
135	intersection points of two of the lines
216	pairs of skew lines
36	double sixes
45	tritangents
120	trihedral pairs, set of six tritangents containing nine lines
40	triples of trihedral pairs, set of 18 tritangents containing all 27 lines
120	triads of azygetic double sixes
270	pairs of syzygetic double sixes

- or: six (like $a_1, a_2, a_3, b_1, b_2, b_3$, which N and N_{123} have in common) which form two triples.

Following Sylvester one speaks accordingly of *syzygetic* and *azygetic* pairs of double sixes. A given double six is syzygetic to 15 others and azygetic to 20 others. A pair of azygetic double sixes form through the 12 lines they do *not* have in common a third double six, which is azygetic to both. There are 120 such triads of azygetic double sixes, and $36 \cdot 15 \div 2 = 270$ pairs of syzygetic double sixes.

We mention here a further geometric curiosity of the 45 planes. Take two of the tritangents which do not meet in a line on the cubic surface, say α_1, α_2 (for example $a_1b_2c_{12}$ and $a_3b_4c_{34}$). These two planes determine three others, denoted $\beta_1, \beta_2, \beta_3$, by the property that each β_i is determined as the tritangent containing each a line of α_1 and of α_2 (in the example above, $a_1b_4, b_2a_3, c_{12}c_{34}$). The third line in each of β_i which is not one of α_i all lie in a common tritangent (for example here $c_{14}c_{23}c_{56}$). Then this is a unique third tritangent denoted α_3 . The set of six tritangents $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)$ have the special property that the nine lines $\alpha_\mu \cap \beta_\nu$ together with a point (19 conditions) determine S . Such a set of six tritangents is called a *trihedral pair*, and there are 120 such; this implies that the equation of the cubic surface S can be written in 120 ways as

$$y_1y_2y_3 + z_1z_2z_3 = 0.$$

One finds the following types:

(20)			(10)			(90)		
a_i	b_j	c_{ij}	c_{il}	c_{jm}	c_{kn}	a_i	b_j	c_{ij}
b_k	c_{jk}	a_j	c_{mn}	c_{ik}	c_{lj}	b_l	a_k	c_{kl}
c_{ik}	a_k	b_i	c_{jk}	c_{ln}	c_{im}	c_{il}	c_{jk}	c_{mn}

(2)

the rows (of each box) giving $\alpha_1, \alpha_2, \alpha_3$ and the columns giving $\beta_1, \beta_2, \beta_3$. This configuration is in some sense complementary to the double sixes: *Starting with nine lines lying in such a trihedral pair, the remaining 18 lines form a unique azygetic triple of double sixes and conversely, the nine lines not contained in a given azygetic triple of double sixes always lie in a trihedral pair*. Furthermore there are 40 *triads (triples) of trihedral pairs*, such that each such triad contains all 27 lines. One is lead to study *enneahedra*, i.e., 9-gons, by which one means sets of 9 tritangents which contain all 27 lines. It turns out that there are two types of such enneahedra: 40 of the first type which have the property that the nine tritangents form *four different* triads of trihedral pairs, and 160 of the second type which belong to a unique triad. We sum up the configuration in Table 2.

The modern point of view is to consider cubic surfaces as del Pezzo surfaces of degree 6. The combinatorics of the 27 lines, as listed in Table 2, are then encoded in the Picard group of the del Pezzo surface. In fact, the complement in $\text{Pic}(S)$ to the hyperplane section, call it $\text{Pic}^0(S)$, is isomorphic to the root lattice of E_6 . The equation of the surface is given by the embedding of S by means of the linear system of elliptic curves through the six given points.

Fix six points in \mathbb{P}^2 , say $x = (p_1, \dots, p_6)$, such that the p_i are in general position, i.e., no three lie on a line, and not all six lie on a conic. Let $\widehat{\mathbb{P}}_x^2$ denote the blow up of \mathbb{P}^2 at all six points, $\varrho_x : \widehat{\mathbb{P}}_x^2 \rightarrow \mathbb{P}^2$. Consider the following curves as classes in $\text{Pic}(\widehat{\mathbb{P}}_x^2)$:

- i) a_1, \dots, a_6 , the exceptional divisors over p_1, \dots, p_6 ;
- ii) b_1, \dots, b_6, b_i the proper transform of the conic q_i passing through all points $p_j, j \neq i$;
- iii) c_{ik} , the proper transform of the line $\overline{p_i p_k}$.

If we consider the surface $\widehat{\mathbb{P}}_x^2$, we have $H^2(\widehat{\mathbb{P}}_x^2, \mathbb{Z}) = [l]\mathbb{Z} \oplus_i \mathbb{Z}a_i$. Let Q be the intersection form on $H^2(\widehat{\mathbb{P}}_x^2, \mathbb{Z})$; then the classes a_i, b_i, c_{ij} fulfill $Q(a_i, a_i) = Q(b_i, b_i) = Q(c_{ij}, c_{ij}) = -1$. In a well-known manner one takes a rank six subset, which is isomorphic to the root lattice of type \mathbf{E}_6 . For details see [DO] and references therein. Consider the orthocomplement of the anti-canonical class on $\widehat{\mathbb{P}}_x^2$, and denote this by $\text{Pic}^0(\widehat{\mathbb{P}}_x^2)$. Recall that the anti-canonical class is $3l + \sum_{i=1}^6 a_i$, and that the anti-canonical embedding of $\widehat{\mathbb{P}}_x^2$ is as a cubic surface. Consequently we may view $\text{Pic}^0(\widehat{\mathbb{P}}_x^2)$ as the orthocomplement of the hyperplane section class of $\text{Pic}(S_x)$, where S_x is the cubic surface which is the anti-canonical embedding. The following elements λ with $Q(\lambda, \lambda) = -2$ form a basis of $\text{Pic}^0(S_x)$:

$$\begin{aligned}
 \alpha_0 &= l - a_1 - a_2 - a_3 \\
 \alpha_1 &= a_1 - a_2 \\
 \alpha_2 &= a_2 - a_3 \\
 \alpha_3 &= a_3 - a_4 \\
 \alpha_4 &= a_4 - a_5 \\
 \alpha_5 &= a_5 - a_6
 \end{aligned}
 \tag{3}$$

These also form a base of a root system of type \mathbf{E}_6 , by taking $\alpha_1, \dots, \alpha_5$ as the sub root system of type \mathbf{A}_5 . Since the classes a_i, b_i, c_{ij} are exceptional, they all represent elements of $\text{Pic}^0(S_x)$, and the 45 tritangents give 45 relations like $a_i + b_j + c_{ij} = 0$. This leads to the following exact sequence of \mathbb{Z} -modules

$$0 \longrightarrow \mathbb{Z}^{24} \longrightarrow \mathbb{Z}^{45} \longrightarrow \mathbb{Z}^{27} \longrightarrow \text{Pic}^0(S_x) \longrightarrow 0
 \tag{4}$$

which we will meet again later.

Acknowledgements: Many of the results in Part II on the geometry of $W(E_6)$ in \mathbb{P}^5 , for example the sequence (4), were attained (several years ago) in collaboration with Duco v. Straten, who unfortunately had to devote himself to more urgent matters. I have also had discussions with W. Barth, Bert v. Geemen, Igor Dolgachev, E. Looijenga, K. Matsumoto, I. Naruki and Masaaki Yoshida on many of the matters presented here, and it is my pleasure to thank them heartily.

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Part I

Projective embeddings of modular varieties

1 The tetrahedron in \mathbb{P}^3

1.1 Arrangements defined by Weyl groups

Let $\Phi(G, T) \subset \mathfrak{t}^*$ be a root system of a simple group G (over \mathbb{C}). Using notations as in Bourbaki we have the roots (for those systems which will be of interest to us in the sequel)

$$\begin{aligned}
\mathbf{A}_n & \pm(\varepsilon_i - \varepsilon_j), 1 \leq i < j \leq n+1; \\
\mathbf{B}_n & \pm(\varepsilon_i \pm \varepsilon_j), \pm\varepsilon_i, 1 \leq i < j \leq n; \\
\mathbf{C}_n & \pm(\varepsilon_i \pm \varepsilon_j), \pm 2\varepsilon_i, 1 \leq i < j \leq n; \\
\mathbf{D}_n & \pm(\varepsilon_i \pm \varepsilon_j), 1 \leq i < j \leq n; \\
\mathbf{F}_4 & \pm(\varepsilon_i \pm \varepsilon_j), \pm\varepsilon_k, \pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4), 1 \leq i < j \leq 4, k = 1, \dots, 4; \\
\mathbf{E}_6 & \pm(\varepsilon_i \pm \varepsilon_j), 1 \leq i < j \leq 5, \pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8), \text{ with an even} \\
& \text{number of “-” signs in the parenthesis;}
\end{aligned} \tag{5}$$

Each root α determines an orthogonal plane α^\perp , and for any arrangement \mathbf{X}_n ,

$$\mathcal{A}(\mathbf{X}_n) := \{\alpha^\perp \mid \alpha \text{ a root}\} \tag{6}$$

is a central arrangement in \mathbb{C}^n , i.e., each of the planes passes through the origin. This induces a projective arrangement in \mathbb{P}^{n-1} , as follows. Blow up the origin of \mathbb{C}^n ; the exceptional divisor is a \mathbb{P}^{n-1} . The *projective arrangement* is the union of the intersections $[H] \cap \mathbb{P}^{n-1}$ in the exceptional divisor, where $[H]$ is the proper transform of the hyperplane $H = \alpha^\perp$ under the blow up. The projective arrangements for \mathbf{B}_n and \mathbf{C}_n coincide, and these arrangements are given in \mathbb{P}^{n-1} with projective coordinates $(x_1 : \dots : x_n)$ as follows:

$$\begin{aligned}
\mathcal{A}(\mathbf{A}_n): & \{x_i = 0, i = 1, \dots, n; x_i = x_j, 1 \leq i < j \leq n\}; \\
\mathcal{A}(\mathbf{B}_n): & \{x_i = 0, i = 1, \dots, n; x_i = \pm x_j, 1 \leq i < j \leq n\}; \\
\mathcal{A}(\mathbf{D}_n): & \{x_i = \pm x_j, 1 \leq i < j \leq n\}; \\
\mathcal{A}(\mathbf{F}_4): & \{x_i = 0, i = 1, \dots, n; x_i = \pm x_j, 1 \leq i < j \leq 4, \frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4)\}; \\
\mathcal{A}(\mathbf{E}_6): & \{x_i = \pm x_j, 1 \leq i < j \leq 5, \frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 + x_6)\}.
\end{aligned} \tag{7}$$

For the arrangement of type \mathbf{A}_n we have made the coordinate transformation $x_1 = \varepsilon_1 - \varepsilon_{n+1}, \dots, x_n = \varepsilon_n - \varepsilon_{n+1}$, so $x_i - x_j = \varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n$, and for \mathbf{E}_6 we have taken x_6 to replace $x_8 - x_7 - x_6$.

The arrangements above are the arrangements defined by the projective reflection groups $PW(\mathbf{X}_n)$. Each hyperplane is the reflection plane for the reflection on the corresponding root. From this point of view these arrangements are studied in [OS].

1.2 Rank 4 arrangements

As described above, the groups $W(\mathbf{A}_4)$, $W(\mathbf{B}_4)$, $W(\mathbf{D}_4)$ and $W(\mathbf{F}_4)$ give rise to projective arrangements in \mathbb{P}^3 . They consist of ten, 16, 12 and 24 planes, respectively. They may also be described as follows (see [GS]):

- $\mathcal{A}(\mathbf{A}_4)$: four faces of a tetrahedron plus the six symmetry planes;
 $\mathcal{A}(\mathbf{B}_4)$: six faces of a cube plus the nine symmetry planes plus the plane at infinity;
 $\mathcal{A}(\mathbf{D}_4)$: six faces of a cube plus the six symmetry planes through two edges each, OR: eight faces of an octahedron plus three symmetry planes containing four vertices each plus the plane at infinity;
 $\mathcal{A}(\mathbf{F}_4)$: “desmic figure”: six faces of the cube, eight faces of an inscribed octahedron, nine symmetry planes and the plane at infinity; this is also determined by the regular 24-cell;

The combinatorial description of these arrangements can be encoded in numbers:

$$t_q(j) := \#\{\mathbb{P}^j\text{'s of the arrangement through which } q \text{ of the reflection planes pass}\}. \quad (8)$$

In the case of the above arrangements we have the following data ($t_q := t_q(0)$, the number of points):

$$\begin{aligned}
 \mathcal{A}(\mathbf{A}_4): & \quad t_6 = 5, t_4 = 10; t_3(1) = 10, t_2(1) = 15. \\
 \mathcal{A}(\mathbf{B}_4): & \quad t_9 = 4, t_6 = 8, t_5 = 12, t_4 = 16; t_4(1) = 6, t_3(1) = 16, t_2(1) = 36. \\
 \mathcal{A}(\mathbf{D}_4): & \quad t_6 = 12, t_3 = 12; t_3(1) = 16, t_2(1) = 18. \\
 \mathcal{A}(\mathbf{F}_4): & \quad t_9 = 24, t_4 = 96; t_4(1) = 18, t_3(1) = 32, t_2(1) = 72.
 \end{aligned} \quad (9)$$

Definition 1.1 An arrangement $\mathcal{A} \subset \mathbb{P}^n$ is said to be in (combinatorial) *general position*, if $t_q(j) = 0$ for all $q > n - j$. All \mathbb{P}^j 's $\subset \mathcal{A}$ for which $j > n - q$ holds are the *singularities* of the arrangement. The singularities are *genuine* if they are not the intersection of higher-dimensional singular loci with one of the planes of the arrangement. The union of all genuine singularities is the *singular locus*.

In the above arrangements we have the following singular loci:

$$\begin{aligned}
 \mathcal{A}(\mathbf{A}_4): & \quad \text{five singular points, ten singular lines;} \\
 \mathcal{A}(\mathbf{B}_4): & \quad 12=4+8 \text{ (genuine) singular points, } 22=6+16 \text{ singular lines;} \\
 \mathcal{A}(\mathbf{D}_4): & \quad 12 \text{ singular points, } 16 \text{ singular lines;} \\
 \mathcal{A}(\mathbf{F}_4): & \quad 24 \text{ (genuine) singular points, } 50=18+32 \text{ singular lines.}
 \end{aligned} \quad (10)$$

1.3 The tetrahedron

Consider now the arrangement $\mathcal{A}(\mathbf{A}_4)$ in (9). By (10) the singular locus consists of five points and ten lines. We introduce the following notation: $P_1 = (1, 0, 0, 0)$, $P_2 = (0, 1, 0, 0)$, $P_3 = (0, 0, 1, 0)$, $P_4 = (0, 0, 0, 1)$, $P_5 = (1, 1, 1, 1)$, and l_{ij} will denote the line joining P_i and P_j . Each line contains two of the five points, and at each of the points four of the ten lines meet. The arrangement is *resolved* by performing the following birational modification of \mathbb{P}^3 :

- a) Blow up the five points, $\varrho_1 : \widehat{\mathbb{P}}^3 \longrightarrow \mathbb{P}^3$;
- b) Blow up the proper transforms of the ten lines, $\varrho_2 : \widetilde{\mathbb{P}}^3 \longrightarrow \widehat{\mathbb{P}}^3$, $\varrho : \widetilde{\mathbb{P}}^3 \longrightarrow \mathbb{P}^3$.

In the resolution 15 exceptional divisors E_1, \dots, E_5 and L_{12}, \dots, L_{45} are introduced. The E_i are the proper transforms of the exceptional divisors introduced under ϱ_1 , and are isomorphic to \mathbb{P}^2 blown up in the four points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(1 : 1 : 1)$, as are the proper transforms H_i of the ten planes of the arrangement. The ten exceptional divisors L_{ij} are each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The symmetry group of $\widetilde{\mathbb{P}}^3$ consists of projective linear transformations of \mathbb{P}^3 which preserve the arrangement $\mathcal{A}(\mathbf{A}_4)$, together with certain *birational* transformations of \mathbb{P}^3 which are *regular* on $\widetilde{\mathbb{P}}^3$, i.e., which contain the singular locus (10) with simple multiplicity in their ramification locus. Hence the Weyl group itself, $W(\mathbf{A}_4) = \Sigma_5$ (symmetric group on five letters) is contained in the symmetry group. But in fact, Σ_6 is the symmetry group, and the extra generator is a permutation of one of the E_i and H_j , which clearly can be done *on* $\widetilde{\mathbb{P}}^3$.

1.4 A birational transformation

Note that since each of the ten lines in (10) contains two of the five points which are blown up under ϱ_1 , the normal bundle of the proper transform of each line on $\widehat{\mathbb{P}}^3$ is $\mathcal{O}(1-2) \oplus \mathcal{O}(1-2) = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. By general results of threefold birational geometry, it follows that

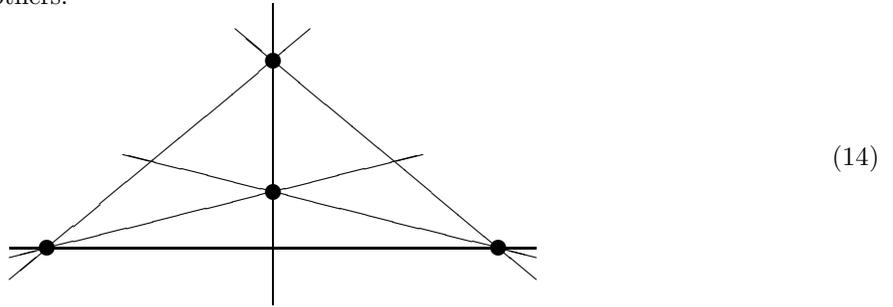
- a) The divisors L_{ij} on $\widehat{\mathbb{P}}^3$ may be blown down to an ordinary threefold rational point (node), i.e., a singularity given by the equation $x^2 + y^2 + z^2 + t^2 = 0$, OR: (12)
- b) The ten lines on $\widehat{\mathbb{P}}^3$ may be blown down to the nodes mentioned in a).

In other words, there is a threefold which we denote by T , which contains ten threefold nodes, with a birational triangle:

$$\begin{array}{ccc}
 \widetilde{\mathbb{P}}^3 & \xrightarrow{\varrho_2} & \widehat{\mathbb{P}}^3 \xrightarrow{\varrho_1} \mathbb{P}^3 \\
 \Pi_2 \searrow & & \swarrow \Pi_1 \\
 & T &
 \end{array}
 \tag{13}$$

The map Π_2 blows down the union of ten disjoint “quadric surfaces” (i.e., divisors isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) to ordinary nodes, while ϱ_2 blows these quadric surfaces down to ten disjoint lines, which Π_1 then blows down to the same ten isolated nodes. The 5+10 divisors E_i and H_j on $\widehat{\mathbb{P}}^3$ have the following properties:

- a) Each is isomorphic to \mathbb{P}^2 blown up in four points;
- b) Each contains ten lines of intersection with the other 15, forming an arrangement in the blown up \mathbb{P}^2 of ten lines meeting in 15 points.
- c) Under the birational map Π_2 each of the divisors E_i and H_j are blown down to a \mathbb{P}^2 ; the image of the ten lines of b) lie four at a time in each of these \mathbb{P}^2 's, as the four t_3 -points of the following arrangement, which is the union of the intersections of the given \mathbb{P}^2 with the others:



- d) The composition $\Pi_2 \circ \varrho_1^{-1}$ restricted to each of the planes H_j is a usual Cremona transformation, blowing up three non-colinear points and blowing down the three joining lines. *Proof:* Take a face H_j of the tetrahedron; ϱ_1 blows up the three vertices it contains, so $\varrho_1^{-1}(H_j)$ (the proper transform of H_j) is \mathbb{P}^2 blown up in three points. Under ϱ_2 , a fourth point of $\varrho_1^{-1}(H_j)$ is blown up, but it is blown down again under Π_2 , as are the proper transforms of the three lines (in the plane H_j) joining the three vertices. By symmetry the same holds for all the H_j .

It follows that on T , the images $\widetilde{H}_j = \Pi_2(H_j)$ and $\widetilde{E}_i = \Pi_2(E_i)$ are copies of \mathbb{P}^2 , each containing four of the ten nodes of T . Furthermore, in each of \widetilde{H}_j and \widetilde{E}_i we have the four t_3 -points of the arrangement (14), which are these four nodes of T . Finally, since there are 15 \mathbb{P}^2 's, ten nodes and four of them in each of the 15 \mathbb{P}^2 's, there are five of these divisors passing through a given node. Explicitly, take the node n_{ij} corresponding to the line l_{ij} in (10). Then it meets the exceptional divisors $\widetilde{E}_i, \widetilde{E}_j$, as well as the three of the \widetilde{H}_ν for which H_ν contains the line l_{ij} .

1.5 Fermat covers associated with arrangements

Let $\mathcal{A} \subset \mathbb{P}^n$ be an arrangement of hyperplanes, i.e., a union $\mathcal{A} = \cup_{i=1}^k H_i$ of k hyperplanes, and let $d \geq 2$ be an integer. To the pair (\mathcal{A}, d) there is an associated function field $\mathcal{L}(\mathcal{A}, d)$, an algebraic extension of the rational function field $\mathcal{M}(\mathbb{P}^n)$. It defines, in a unique way, a branched cover $Y(\mathcal{A}, d) \rightarrow \mathbb{P}^n$, and a unique desingularisation $\tilde{Y}(\mathcal{A}, d)$. The function field is defined by:

$$\mathcal{L}(\mathcal{A}, d) = \mathbb{C} \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) \left[\sqrt[d]{H_2/H_1}, \dots, \sqrt[d]{H_k/H_1} \right], \quad (15)$$

and the cover $Y(\mathcal{A}, d)$ is the so-called Fox closure of the étale cover over $\mathbb{P}^n - \mathcal{A}$ which is defined by (15). $Y(\mathcal{A}, d)$ is smooth outside of the *singular locus* of \mathcal{A} (Definition 1.1), and the singularities of $Y(\mathcal{A}, d)$ are resolved by resolving the singularities of \mathcal{A} . This is done by first blowing up all (genuine, i.e., not near pencil) singular points, then all singular lines, and so forth. The resolution (11) is a typical example. This is described in more detail in the author's thesis; the desingularisation $\tilde{Y}(\mathcal{A}, d)$ is the fibre product in the following diagram:

$$\begin{array}{ccc} \tilde{Y}(\mathcal{A}, d) & \longrightarrow & Y(\mathcal{A}, d) \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \tilde{\mathbb{P}}^n & \xrightarrow{\varrho} & \mathbb{P}^n \end{array} \quad (16)$$

where the horizontal arrows are modifications and the vertical arrows are Galois covers with Galois group $(\mathbb{Z}/d\mathbb{Z})^{k-1}$, which is the Galois group of the field extension of (15). ϱ is the modification of \mathbb{P}^n which resolves the singularities of \mathcal{A} . For example, each singular point P on $Y(\mathcal{A}, d)$ is resolved by an *irreducible* divisor D_P , which itself is a Fermat cover $Y(\mathcal{A}', d)$, where $\mathcal{A}' \subset \mathbb{P}^{n-1}$ is the arrangement induced in the exceptional \mathbb{P}^{n-1} which resolves the point $P' = \pi(P)$. It consists of k' planes, where k' is the number of the k hyperplanes which meet at the point P' , and in the process of resolving $Y(\mathcal{A}, d)$, the cover $Y(\mathcal{A}', d)$ is resolved also. Hence on $\tilde{Y}(\mathcal{A}, d)$ there is a *smooth* divisor D_P which resolves the singular point P of $Y(\mathcal{A}, d)$.

The singular covers $Y(\mathcal{A}, d)$ can also be realised as complete intersections, namely the intersections of $N = k - n$ Fermat hypersurfaces in \mathbb{P}^{k-1} :

$$\begin{aligned} F_1 &= a_{11}x_1^d + \dots + a_{1k}x_k^d \\ &\vdots \\ F_N &= a_{N1}x_1^d + \dots + a_{Nk}x_k^d \end{aligned} \quad (17)$$

where $a_{11}H_1 + \dots + a_{1k}H_k, \dots, a_{N1}H_1 + \dots + a_{Nk}H_k$ are the $(k - n)$ linear relations among the k hyperplanes H_i . The map $Y(\mathcal{A}, d) \rightarrow \mathbb{P}^n$ is realised explicitly by the map $(x_1, \dots, x_k) \mapsto (x_1^d, \dots, x_k^d)$.

1.6 The hypergeometric differential equation

The Fermat covers for the arrangement $\mathcal{A}(\mathbf{A}_4)$ are closely related to solutions of the hypergeometric differential equation on \mathbb{P}^3 , which is an algebraic differential equation with regular singular points, whose singular locus *coincides* with the arrangement $\mathcal{A}(\mathbf{A}_4)$, meaning that solutions are locally branched along the planes of the arrangement.

First we introduce a new notation for the 15 surfaces E_i, H_j . These can be numbered by pairs (i, j) , $i < j \in \{0, \dots, 5\}$, with $E_i = H_{0i}$ and

$$H_{ij} \cap H_{kl} \neq \emptyset \iff i \neq j \neq k \neq l. \quad (18)$$

We denote by $0i$ the point P_i in \mathbb{P}^3 , and by $0ij$ the singular line joining $0i$ and $0j$ in \mathbb{P}^3 . We then let L_{0ij} denote the exceptional divisor on $\tilde{\mathbb{P}}^3$. We have (in \mathbb{P}^3)

$$H_{ij} \cap H_{kl} = 0mn \iff \{i, j, k, l\} \cap \{0, m, n\} = \emptyset. \quad (19)$$

We want to consider branched covers $Y \rightarrow \tilde{\mathbb{P}}^3$ (with $\tilde{\mathbb{P}}^3$ as in (11)), which are branched along the H_{ij} and the L_{0ij} . Hence we let

$$n_{ij} := \text{branching degree along } H_{ij}; \quad n_{0ij} := \text{branching degree along } L_{0ij}, \quad (20)$$

and of course $n_{ij}, n_{0ij} \in \mathbb{Z} \cup \infty$. (It makes sense to allow negative branching degrees, as we will see below.)

To define the hypergeometric differential equation we may just as well work on \mathbb{P}^n with homogenous coordinates $(x_0 : \dots : x_n)$, and consider the arrangement $\mathcal{A}(\mathbf{A}_n)$ of (7). Let $\lambda_i \in \mathbb{Q}$, $i = 0, \dots, n+1, \infty$, $\sum_i \lambda_i = n+1$. The hypergeometric differential equation is:

$$\begin{cases} (x_i - x_j)\partial_i\partial_j F + (\lambda_i - 1)(\partial_i F - \partial_j F) = 0, & 1 \leq i < j \leq n \\ x_i(x_i - 1)\partial_i^2 F + P_i(x, \lambda)\partial_i F + (\lambda_i - 1) \sum \frac{x_\alpha(x_\alpha - 1)}{(x_i - x_\alpha)} \partial_\alpha F + \lambda_\infty(1 - \lambda_i)F = 0, & 1 \leq i \leq n. \end{cases} \quad (21)$$

where

$$P_i(x, \lambda) = x_i(x_i - 1) \sum \frac{1 - \lambda_\alpha}{x_i - x_\alpha} + \lambda_0 + \lambda_i - 3 - (2\lambda_i + \lambda_0 + \lambda_{n+1})x_i.$$

A solution of (21) turns out to be a period of an algebraic curve (the periods are many valued, as are the solutions of (21)). The curve is

$$y^\nu = x^{\mu_0}(x-1)^{\mu_{n+1}}(x-t_1)^{\mu_1} \cdots (x-t_n)^{\mu_n}, \quad (22)$$

where the μ_i, ν are related to the λ_i by the relation

$$\frac{\mu_i}{\nu} = 1 - \lambda_i. \quad (23)$$

The equation (21) has an $(n+1)$ -dimensional solution space, spanned by $(n+1)$ periods of differentials of the curve (22):

$$\omega_i = \int_{\gamma_i} \frac{dx}{y}, \quad \langle \gamma_0, \dots, \gamma_n \rangle = H^1(C, \mathbb{Z}). \quad (24)$$

Taking these gives a homogenous many valued map

$$(\omega_0, \dots, \omega_n) : D \subset \widetilde{\mathbb{P}}^n \xrightarrow{\phi} \mathbb{P}^n, \quad (25)$$

where D is some Zariski open set (see (27) below). The map is well-defined, since not all ω_i vanish simultaneously. For very special values of the parameters λ_i , the image of ϕ is the complex ball $\mathbb{B}_n \subset \mathbb{P}^n$ (this is just the Borel embedding of \mathbb{B}_n in its compact dual). In fact, one has the following theorem:

Theorem 1.2 ([DM], [T]) *If the following conditions are satisfied, then $\phi(D) = \mathbb{B}_n$:*

$$\sum \mu_i = 2, \quad \forall_{i,j} : (1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z} \cup \infty.$$

In this case there exists a finite cover

$$Y \longrightarrow D$$

branched along the total transform of $\mathcal{A}(\mathbf{A}_n)$, which is a quotient $\Gamma \backslash \mathbb{B}_n$ with Γ torsion free.

The integers $n_{ij} := (1 - \mu_i - \mu_j)^{-1}$ are then just the branching degrees of $Y \longrightarrow D$ along the divisor H_{ij} . In fact the numbering introduced in (18) can be done analogously for any n .

In the special case of $\mathcal{A}(\mathbf{A}_4)$ on \mathbb{P}^3 , the integers n_{0ij} of (20) are determined by the relation

$$n_{0ij} = 2 \left(\frac{1}{n_{kl}} + \frac{1}{n_{lm}} + \frac{1}{n_{km}} \right)^{-1},$$

where the line $0ij$ is the intersection of H_{kl}, H_{lm}, H_{km} , and these together with the n_{ij} describe the branching degrees along the entire branch locus. The solutions of the equations in Theorem 1.2 are as follows:

$$\begin{aligned} 1) & \quad \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \\ 2) & \quad \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \\ 3) & \quad \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \\ 4) & \quad \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \\ 5) & \quad \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \\ 6) & \quad \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \\ 7) & \quad \frac{7}{12}, \frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \end{aligned} \quad (26)$$

The set D of (25) is determined as the complement of

$$H_\infty = \{H_{ij} \mid n_{ij} = \infty; L_{0ij} \mid n_{0ij} = \infty\} \subset \tilde{\mathbb{P}}^3. \quad (27)$$

This is the locus which the uniformizing map (25) maps onto the *boundary* of $\mathbb{B}_3 \subset \mathbb{P}^3$, i.e., $\phi(D) = \mathbb{B}_n$, $\phi(H_\infty) \subset \partial\mathbb{B}_n$. This requires of course that the corresponding covers of the divisors on the cover \tilde{Y} be abelian varieties (as these are the compactification divisors on ball quotients). This can happen as follows

- (i) On one of the H_{ij} , this can occur if the branching degrees are: 2 for the six lines of (14), and -4 for the four exceptional curves. (28)
- (ii) On L_{0ij} , this can happen if $\mu_k + \mu_l + \mu_m = 1$, $\mu_0 + \mu_i + \mu_j = 1$.

In the second case, the surface S_{0ij} covering L_{0ij} is of the form $C_1 \times C_2$, where $C_1 \rightarrow \mathbb{P}^1$ (respectively $C_2 \rightarrow \mathbb{P}^1$) is a cover, with branching determined by (μ_k, μ_l, μ_m) (respectively determined by (μ_0, μ_i, μ_j)). It is an abelian variety \iff both curves C_i are elliptic. Note that $Y \rightarrow \tilde{\mathbb{P}}^3$ will be a Fermat cover \iff all n_{ij} coincide \iff all μ_i conicide. In particular,

Proposition 1.3 *The only ball quotient in the list (26) which is a Fermat cover which is a ball quotient is the solution 1), namely $Y(\mathcal{A}(\mathbf{A}_4), 3)$ is a ball quotient.*

Remark 1.4 We will see later (see I3 following Lemma 9.2 below) that the solution 4) gives rise also to a Fermat cover which is a ball quotient, namely $Y(\mathcal{A}(\mathbf{D}_4), 3)$.

2 The Segre cubic \mathcal{S}_3

In this section we will show that the variety T of (13) has a projective embedding as a cubic hypersurface known as the Segre cubic, which we denote by \mathcal{S}_3 .

2.1 Segre's cubic primal

In \mathbb{P}^5 with homogenous coordinates $(x_0 : \dots : x_5)$ consider the locus

$$\mathcal{S}_3 := \left\{ \sum_{i=0}^5 x_i = 0; \sum_{i=0}^5 x_i^3 = 0 \right\}. \quad (29)$$

As the first equation is linear, this shows that \mathcal{S}_3 is a hypersurface, i.e., $\mathcal{S}_3 \subset \mathbb{P}^4 = \{x \in \mathbb{P}^5 \mid \sum x_i = 0\}$. Using $(x_0 : \dots : x_5)$ as projective coordinates, the relation $x_5 = -x_0 - \dots - x_4$ gives the equation of \mathcal{S}_3 as a hypersurface; however, the equation in \mathbb{P}^5 shows that \mathcal{S}_3 is invariant under the symmetry group Σ_6 , acting on \mathbb{P}^5 by permuting coordinates, which is not so immediate from the hypersurface equation.

It is known that for any degree d there is an upper bound on the number of ordinary double points which a hypersurface of degree d can have, the so-called Varchenko bound. For cubic threefolds this number is ten, and it has been known since the last century that \mathcal{S}_3 is the *unique* (up to isomorphism) cubic with ten nodes. The nodes on \mathcal{S}_3 are given by the points of \mathbb{P}^5 for which three of the coordinates are 1 and the other three are -1 . This is just the Σ_6 -orbit of

$$(1, 1, 1, -1, -1, -1). \quad (30)$$

There is another interesting locus on \mathcal{S}_3 . Consider, in \mathbb{P}^5 , the planes P_σ given by

$$P_\sigma = \{x_{\sigma(0)} + x_{\sigma(3)} = x_{\sigma(1)} + x_{\sigma(4)} = x_{\sigma(2)} + x_{\sigma(5)} = 0\}, \quad (31)$$

where $\sigma \in \Sigma_6$. There are 15 such P_σ 's, the Σ_6 -orbit of

$$P_{id} = \{x_0 + x_3 = x_1 + x_4 = x_2 + x_5 = 0\}. \quad (32)$$

One checks easily that each P_σ contains four of the double points; for example P_{id} contains the following:

$$(1, 1, -1, 1, -1, -1), (1, -1, 1, 1, -1, -1), (1, -1, -1, 1, 1, -1), (1, -1, -1, 1, -1, 1).$$

Furthermore, the intersection of P_{id} with the other P_σ is the line arrangement (14). It is easily checked that each P_σ is contained entirely in \mathcal{S}_3 . One can also argue as follows. Any line in \mathbb{P}^5 which contains two of the nodes of \mathcal{S}_3 meets \mathcal{S}_3 with multiplicity 4, hence is contained in \mathcal{S}_3 . Similarly, each P_σ meets \mathcal{S}_3 in the six lines of the arrangement (14), hence is contained in \mathcal{S}_3 .

We just remark that the hyperplane sections $\{x_i = 0\}$ of \mathcal{S}_3 are cubic surfaces with equation

$$\mathcal{S}_3 = \left\{ \sum_{i=0}^4 x_i = \sum_{i=0}^4 x_i^3 = 0 \right\}. \quad (33)$$

This cubic surface is known as the Clebsch diagonal surface and is a remarkably beautiful object. It is the unique cubic surface having Σ_5 as symmetry group. The relation between \mathcal{S}_3 and the icosahedral group was studied by Hirzebruch. It turns out that \mathcal{S}_3 is A_5 -equivariantly birational to the Hilbert modular surface for $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$, of level $\sqrt{5}$.

Other interesting hyperplane sections are given by the hyperplanes $\mathcal{T}_{ij} = \{x_i - x_j = 0\}$; indeed, \mathcal{T}_{ij} also contains four of the ten nodes, hence $\mathcal{T}_{ij} \cap \mathcal{S}_3$ is a four-nodal cubic surface. This four-nodal cubic surface is projectively unique, and is called the Cayley cubic.

2.2 A birational transformation

Theorem 2.1 *The variety T of equation (13) is biregular to \mathcal{S}_3 ; the isomorphism $\psi : T \rightarrow \mathcal{S}_3$ defined below is Σ_6 -equivariant.*

Proof: Following Baker [B], IV, p. 152, we define a birational map

$$\beta : \mathbb{P}^3 \dashrightarrow \mathcal{S}_3.$$

Consider all quadric surfaces in \mathbb{P}^3 passing through the points P_i of (10). A base of this linear system is given by the following degenerate quadrics. Let $(z_0 : \dots : z_3)$ be homogenous coordinates on \mathbb{P}^3 , and set

$$\begin{aligned} \xi &= z_0(z_3 - z_1), & \eta &= z_1(z_3 - z_2), & \zeta &= z_2(z_3 - z_0); \\ \xi' &= z_1(z_3 - z_0), & \eta' &= z_2(z_3 - z_1), & \zeta' &= z_0(z_3 - z_2). \end{aligned} \quad (34)$$

These quadrics satisfy the relations $\xi + \eta + \zeta = \xi' + \eta' + \zeta'$ and $\xi\eta\zeta = \xi'\eta'\zeta'$. Now change coordinates by setting

$$\begin{aligned} \xi &= X + Y, & \eta &= Y + Z, & \zeta &= X + Z; \\ \xi' &= -(X' + Y'), & \eta' &= -(Y' + Z'), & \zeta' &= -(X' + Z'). \end{aligned} \quad (35)$$

Then the relations $\xi + \eta + \zeta = \xi' + \eta' + \zeta'$ and $\xi\eta\zeta = \xi'\eta'\zeta'$ become

$$\begin{aligned} X + Y + Z + X' + Y' + Z' &= 0 \\ X^3 + Y^3 + Z^3 + (X')^3 + (Y')^3 + (Z')^3 &= 0. \end{aligned} \quad (36)$$

One sees this is just equation (29) of the Segre cubic. This yields a rational map $\beta : \mathbb{P}^3 \dashrightarrow \mathcal{S}_3$, $\beta(z_0 : z_1 : z_2 : z_3) = (X, Y, Z, X', Y', Z')$. The base locus of the linear system of quadrics defining β (34) is the five points of (10), as the quadrics all contain these points. It follows that β blows up all five points, the exceptional divisors E_1, \dots, E_5 being projective planes. Now consider one of the ten lines of (10); for example, the one given by $z_2 = z_3 = 0$. Then $\eta = \zeta = \eta' = \zeta' = 0$ and $\xi = \xi' = -z_0 z_1$. In other words, β maps that line to the point $(1, 0, 0, 1, 0, 0)$ in the $(\xi, \eta, \zeta, \xi', \eta', \zeta')$ space, which is the point $(1, 1, -1, -1, -1, 1)$ in the (X, Y, Z, X', Y', Z') space. But that is just one of the ten nodes of \mathcal{S}_3 . From Σ_6 -symmetry we conclude that $\beta : \mathbb{P}^3 \dashrightarrow \mathcal{S}_3$

coincides with the map $\beta = \Pi = \Pi_1 \circ \varrho_1^{-1}$, with Π_1 as in (13) and ϱ_1 as in (11). In other words, $\beta = \Pi$ is the composition of morphisms

$$\begin{array}{ccc} & \widehat{\mathbb{P}}^3 & \\ \varrho_1 \swarrow & & \searrow \Pi_2 \\ \mathbb{P}^3 & \xrightarrow{\quad \Pi \quad} & T, \end{array} \quad (37)$$

and since (36) states that $\beta = \Pi$ maps onto \mathcal{S}_3 , this gives an isomorphism $T \cong \mathcal{S}_3$. Explicitly, $t \in T$, $t \mapsto (\varrho_1 \circ \Pi_2^{-1})(t) \mapsto \beta((\varrho_1 \circ \Pi_2^{-1})(t)) = \psi(t) \in \mathcal{S}_3$ is the desired map. The Σ_6 -equivariance follows from the fact that the whole diagram (37) is Σ_6 -equivariant. \square

Now consider the Picard group of \mathcal{S}_3 . From the explicit form of birational map as given by Theorem 2.1 and (13), we see that $\text{Pic}(\mathcal{S}_3)$ is generated by the image of the hyperplane class, call it H , and the five exceptional classes E_i . It follows that $\text{Pic}(\mathcal{S}_3)$ has rank 6, and the primitive part $\text{Pic}^0(\mathcal{S}_3)$, i.e., the complement of the hyperplane class, has rank 5. The 15 classes H_{ij} introduced in (18) (these are the 15 linear \mathbb{P}^2 's on \mathcal{S}_3 noted in (31)) give classes in $\text{Pic}(\mathcal{S}_3)$ and in $\text{Pic}^0(\mathcal{S}_3)$. The 15 hyperplanes

$$\mathcal{H}_{ij} = \{x_i + x_j = 0\}, \quad (38)$$

each of which meets \mathcal{S}_3 in three of the 15 \mathbb{P}^2 's, give 15 *relations* in $\text{Pic}^0(\mathcal{S}_3)$: since $\mathcal{H}_{ij} \cap \mathcal{S}_3$ is a hyperplane section, the sum of the three \mathbb{P}^2 's cut out by \mathcal{H}_{ij} , i.e., $H_{i_1, j_1} + H_{i_2, j_2} + H_{i_3, j_3} = \mathcal{H}_{ij} \cap \mathcal{S}_3$, is linearly equivalent to the hyperplane class. This yields the following exact sequence of \mathbb{Z} -modules:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & \mathbb{Z}\{\mathcal{H}_{ij}\} & \longrightarrow & \mathbb{Z}\{H_{ij}\} & \longrightarrow & \text{Pic}^0(\mathcal{S}_3) & \longrightarrow & 1 \\ & & \parallel \wr & & \parallel \wr & & \parallel \wr & & \parallel \wr & & \\ 1 & \longrightarrow & \mathbb{Z}^5 & \longrightarrow & \mathbb{Z}^{15} & \longrightarrow & \mathbb{Z}^{15} & \longrightarrow & \mathbb{Z}^5 & \longrightarrow & 1. \end{array} \quad (39)$$

Lemma 2.2 *In the sequence (39), all \mathbb{Z} -modules are Σ_6 -modules, i.e., the exact sequence is one of Σ_6 -modules.*

Proof: This is visible for the right three entries of the first sequence in (39), and it then follows for K . \square

Now consider a generic hyperplane section of \mathcal{S}_3 ; this is a smooth cubic surface. Let $\nu : S = \mathcal{S}_3 \cap H \hookrightarrow \mathcal{S}_3$ denote the inclusion of the section, and let $\nu^* : H^2(\mathcal{S}_3, \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$ be the induced map on cohomology. Then by the Lefschetz hyperplane theorem, this map is *injective*, and since both S and \mathcal{S}_3 are regular (i.e., not irregular, that is, have no holomorphic one forms), we may view this as an injective map of the Picard groups: $\text{Pic}(\mathcal{S}_3) \hookrightarrow \text{Pic}(S)$, and a corresponding inclusion on the primitive part. Recall also that we have on the cubic surface 27 generators (the 27 lines), 45 relations among these (the 45 tritangents), and an exact sequence on $\text{Pic}^0(S)$ as in (4). All in all we get the following map of sequences as in (39):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^5 & \longrightarrow & \mathbb{Z}^{15} & \longrightarrow & \mathbb{Z}^{15} & \longrightarrow & \mathbb{Z}^5 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}^{24} & \longrightarrow & \mathbb{Z}^{45} & \longrightarrow & \mathbb{Z}^{27} & \longrightarrow & \mathbb{Z}^6 & \longrightarrow & 1. \end{array} \quad (40)$$

where the right hand groups are $\text{Pic}^0(\mathcal{S}_3)$ and $\text{Pic}^0(S)$, respectively, and the down arrows are inclusions (by Lefschetz). Note that this corresponds to a symmetry breaking. Indeed, on the first sequence there is a symmetry group Σ_6 acting, as already noted, while on the group $\text{Pic}^0(S)$, in fact on the whole second sequence, the group $W(E_6)$ acts naturally, as is well-known.

Proposition 2.3 *The ideal $\mathcal{J}(10)$ of the ten nodes is generated by the five quadrics \mathcal{R}_λ of the Jacobian ideal of \mathcal{S}_3 .*

Proof: The inclusion $\text{Jac}(\mathcal{S}_3) \subset \mathcal{J}(10)$ is obvious, and the five elements of $\text{Jac}(\mathcal{S}_3)$ are clearly independent. The fact that $\mathcal{J}(10)$ has rank 5 has been verified by standard basis computations (with the algebra program Macaulay). \square

Corollary 2.4 *The ideal of the ten nodes of \mathcal{S}_3 , $\mathcal{J}(10)$, is the Jacobian ideal of \mathcal{S}_3 .* \square

2.3 Uniformisation

In this section we will show that the Segre cubic \mathcal{S}_3 is actually the Satake compactification of a Picard modular variety. Let $K = \mathbb{Q}(\sqrt{-3})$ be the field of Eisenstein numbers, and consider the \mathbb{Q} -group $G = U(3, 1; K)$, the unitary group of a hermitian form on a four-dimensional K -vector space with signature $(3, 1)$. Consider the arithmetic group $\Gamma := G_{\mathbb{Z}} = U(3, 1; \mathcal{O}_K) \subset G(K)$, where \mathcal{O}_K denotes the ring of integers in K . It acts on the three-ball with non-compact quotient X_{Γ} . Consider the principal congruence subgroups $\Gamma(\sqrt{-3})$ and $\Gamma(3)$, as defined in [J]. These determine a corresponding level structure in the sense of Definition 2.5 of [J]. Now note the following well-known isomorphisms:

$$\Gamma/\Gamma(\sqrt{-3}) = \Sigma_6, \quad \Gamma(3)/\Gamma(\sqrt{-3}) = (\mathbb{Z}/3\mathbb{Z})^9. \tag{41}$$

It follows from this that the corresponding quotients $X(a) := X_{\Gamma(a)}$, $a = 1, \sqrt{-3}, 3$, yield Galois covers

$$X(3) \xrightarrow{(\mathbb{Z}/3\mathbb{Z})^9} X(\sqrt{-3}) \xrightarrow{\Sigma_6} X(1), \tag{42}$$

which explicitly describe the level structures involved. As usual let $X(a)^*$ denote the Satake compactification.

Theorem 2.5 *There is a commutative diagram*

$$\begin{array}{ccccc}
 & & \overline{X}(3) & & \\
 & \swarrow & \downarrow & \searrow & \\
 \tilde{Y} & & & & X(3)^* \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 & Y^{\wedge} & & & \\
 & \downarrow & & & \\
 & \overline{X}(\sqrt{-3}) & & & \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 \tilde{\mathbb{P}}^3 & & & & X(\sqrt{-3})^* \\
 & \swarrow & \downarrow & \swarrow & \\
 & \mathcal{S}_3 & & &
 \end{array} \tag{43}$$

where the horizontal maps from right to left are isomorphisms, those from left to right are birational, and the vertical maps are $(\mathbb{Z}/3\mathbb{Z})^9$ covers.

Proof: First we have, over \mathbb{P}^3 , a singular cover T_{DM} defined by the solution 1) of (26). This is desingularised by blowing up the \mathbb{P}^3 along the singular locus of the arrangement, $\tilde{\mathbb{P}}^3 \leftarrow \tilde{T}_{DM}$. From the fact that all $\mu_i = 1/3$, we see that all n_{ij} and n_{0ij} are equal to three, that is T_{DM} is the Fermat cover $Y(\mathcal{A}(\mathbf{A}_4), 3)$, and \tilde{T}_{DM} is its desingularisation $\tilde{Y} := \tilde{Y}(\mathcal{A}(\mathbf{A}_4), 3)$ as in (16); see also Proposition 1.3. By Theorems 1.2 and 2.1, \tilde{Y} is the desingularisation of the ball quotient $\Gamma' \backslash \mathbb{B}_3$, for some torsion free group Γ' . Blowing \tilde{Y} down from $\tilde{\mathbb{P}}^3$ to \mathcal{S}_3 gives the singular variety Y^{\wedge} , which we will see in a minute is the Satake compactification of the ball quotient. Hence we only need to identify the groups and check the compactifications coincide. As to the first, we start with

Lemma 2.5.1 *Let $G_{\mathbb{Q}}$ be an isotropic \mathbb{Q} -form of $U(3, 1)$, $G_{\mathbb{Q}} \sim U(3, 1; L)$, L imaginary quadratic over \mathbb{Q} , and let $\Gamma \subset G_{\mathbb{Q}}$ a torsion free arithmetic subgroup with arithmetic quotient X_{Γ} , Baily-Borel compactification X_{Γ}^* and toroidal compactification \overline{X}_{Γ} . Then the isomorphism class of a single compactification divisor determines the field L , and hence $G_{\mathbb{Q}}$ up to isogeny.*

Proof: First note that for $U(3, 1)$ the parabolic (there is only one conjugacy class of parabolics, as the \mathbb{R} -rank is one) takes on the particularly simple form

$$\begin{aligned} P &\cong (\mathcal{RK}) \rtimes \mathcal{Z}V, & \mathcal{R} &\cong \mathbb{R}^\times, & \mathcal{K} &= SU(2) \times U(1) \\ & & \mathcal{Z} &= \mathbb{R}, & V &= \mathbb{C}^2 \end{aligned} \quad (44)$$

For the \mathbb{Q} -form of P , it follows that $V_{\mathbb{Q}} \cong L^2$ for some imaginary quadratic field L , and for the arithmetic parabolic $\Gamma_P \subset P$, $\Gamma_P \cap V_{\mathbb{Q}} \subset (\mathcal{O}_L)^2$ is some lattice. Furthermore, the theory of toroidal embeddings shows that a compactification divisor of \overline{X}_{Γ} is of the form $\mathbb{C}^2/(\Gamma_P \cap V_{\mathbb{Q}})$, which has complex multiplication by L , so its isomorphism class determines L , which was to be shown. \square

Now an easy calculation shows what the compactification divisors on \tilde{Y} are. Namely, these are the irreducible components of the inverse image in \tilde{Y} of the exceptional divisors $L_{0ij} \subset \tilde{\mathbb{P}}^3$, $L_{0ij} \cong \mathbb{P}^1 \times \mathbb{P}^1$. The local geometry of the arrangement shows the branch locus in L_{0ij} is of the form $p_1^*(\mathcal{O}(3)) \otimes p_2^*(\mathcal{O}(3))$, i.e., of the form $\{0\} \times \mathbb{P}^1$, $\{1\} \times \mathbb{P}^1$, $\{\infty\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{0\}$, $\mathbb{P}^1 \times \{1\}$, $\mathbb{P}^1 \times \{\infty\}$. It is well-known that the elliptic curve $E \rightarrow \mathbb{P}^1$, branched at $(0, 1, \infty)$ to degree 3, with Galois group $\mathbb{Z}/3\mathbb{Z}$, is the elliptic curve $E_{\varrho} = \mathbb{C}/\mathbb{Z} \oplus \varrho\mathbb{Z}$, $\varrho = e^{2\pi i/3}$. From this it follows

Lemma 2.5.2 *The compactification divisors Δ_i of \tilde{Y} are products*

$$\Delta_i \cong E_{\varrho} \times E_{\varrho},$$

where E_{ϱ} is the unique elliptic curve with $\mathbb{Z}/6\mathbb{Z}$ as automorphism group, i.e., $E_{\varrho} = \mathbb{C}/\mathbb{Z} \oplus \varrho\mathbb{Z} = \{x^3 + y^3 + z^3 = 0\}$ and $\text{Aut}(E_{\varrho}) = \langle \pm 1, \pm \varrho, \pm \varrho^2 \rangle$. \square

Note that the morphism $\tilde{Y} \rightarrow Y^{\wedge}$ blows down the Δ_i to singular points (just as $\tilde{\mathbb{P}}^3 \rightarrow \mathcal{S}_3$ blows down the L_{0ij} to the nodes of \mathcal{S}_3) which lie over the nodes of \mathcal{S}_3 . From this and the well-known fact that the Satake compactification of a ball quotient has only isolated, zero-dimensional singularities, which are resolved in a torus embedding by means of complex tori or quotients thereof, we get the following

Corollary 2.5.1 *The variety Y^{\wedge} is the Satake compactification of the quotient $\Gamma' \backslash \mathbb{B}_3$, with $\Gamma' \subset G_{\mathbb{Q}}$ and $G_{\mathbb{Q}}$ isogenous to $U(3, 1; K)$, K the field of Eisenstein numbers as above. \square*

Now that it is established that Γ' is (isogenous to) an arithmetic subgroup of $U(3, 1; K)$, group-theoretic methods can be applied to determine the arithmetic subgroup. This is done in detail in [J], Lemma 2.9 and Theorem 2.11. The result is: $\Gamma' = P\Gamma(3)$, and the group $\Gamma_{\mathcal{S}_3}$ giving rise to the Segre cubic is $\Gamma_{\mathcal{S}_3} = P\Gamma(\sqrt{-3})$. This yields the statements of the theorem on the arithmetic groups. The compactification divisors of \tilde{Y} coincide by Lemma 2.5.2 with those of $\overline{X}_{\Gamma'}$, and these are blown down under $\tilde{Y} \rightarrow Y^{\wedge}$ to the singularities on the Satake compactification which is Y^{\wedge} , $Y^{\wedge} \cong X_{\Gamma'}^*$. The cover $\tilde{Y} \rightarrow \tilde{\mathbb{P}}^3$ (respectively $Y^{\wedge} \rightarrow \mathcal{S}_3$) is now readily identified with $\overline{X}(3) \rightarrow \overline{X}(\sqrt{-3})$ (respectively with $X(3)^* \rightarrow X(\sqrt{-3})^*$) of (42), from the fact that the branching loci, degrees and group actions coincide. Details can be found in [J]. \square

2.4 Moduli interpretation

Now applying Shimura's theory we get the following moduli description of \mathcal{S}_3 (see [J], §2 for details).

Theorem 2.6 *Any point $x \in \mathcal{S}_3 - \{\text{ten nodes}\}$ determines a unique isomorphism class of principally polarised abelian fourfolds with complex multiplication by $K = \mathbb{Q}(\sqrt{-3})$ and a level $\sqrt{-3}$ structure. The signature of the complex multiplication is $(3, 1)$. Any point $x \in Y^{\wedge} - \{\text{inverse image under } \phi \text{ of (43) of the ten nodes}\}$ determines a unique isomorphy class of abelian fourfolds as above with a level 3 structure.*

Moreover, the moduli interpretation of the 15 \mathbb{P}^2 's on \mathcal{S}_3 is given in [J].

Proposition 2.7 *The 15 \mathbb{P}^2 's on \mathcal{S}_3 are compactifications of two-dimensional ball quotients which are moduli spaces of those abelian fourfolds A_x^4 as above which split:*

$$A_x^4 \cong A_x^3 \times E_{\varrho}.$$

The intersections of the 15 planes determine moduli points of A_x^4 which further decompose, i.e., A_x^3 splits.

Remark 2.8 It is natural to ask whether, given a point $x \in \mathcal{S}_3$, one can give the equations defining the abelian variety A_x occurring in Theorem 2.6. In some sense one can. First it turns out the A_x is the Jacobian of an algebraic curve, as described by the hypergeometric equation as in equation (22). Since the parameters are by Proposition 1.3 the set 1) in (26), these curves have the form:

$$C_\tau = \{y^3 = \prod_{i=1}^6 (x - t_i(\tau))\}; \quad (45)$$

C_τ obviously has an automorphism of order 3, given by $y \mapsto \varrho y$ with the third root of unity ϱ . This yields an automorphism of the Jacobian of C_τ . Without much difficulty one finds

(i) $\text{Jac}(C_\tau) = A_\tau$ has complex multiplication by \mathcal{O}_K , the signature is (3,1).

(ii) The automorphism group is \mathcal{O}_K^* , and is given by multiplication by $\pm\varrho$ in \mathcal{O}_K .

The most direct way to see this is to write down the Jacobian of the curve (45) and show that its periods have the complex multiplication. A basis of the (1,0) differentials on C_τ written in the normal form

$$y^3 = x(x-1)(x-t_1)(x-t_2)(x-t_3) \quad (46)$$

is given by

$$\int \frac{dx}{\sqrt[3]{x(x-1)(x-t_1)(x-t_2)(x-t_3)}}; \quad (47)$$

choosing a base of $H_1(C_\tau, \mathbb{Z})$ and taking the integrals over the elements of that base gives the Jacobian; the multiplication by ϱ is then evident. Hence one may invoke Shimura's theory to conclude:

Lemma 2.9 *The isomorphism classes of the Jacobians of the curves (46) are given as the points of the arithmetic quotient $PU(3, 1; \mathcal{O}_K) \backslash \mathbb{B}_3$. Putting a $\sqrt{-3}$ level structure on the Jacobians yields the moduli space $\Gamma(\sqrt{-3}) \backslash \mathbb{B}_3$.*

The latter space has already been identified with the open subset of smooth points on \mathcal{S}_3 .

The precise relation between the moduli *point* $\tau \in \mathbb{B}_3$ and the *values* of the t_i has been derived for surfaces, i.e., for τ in one of the subballs covering one of the 15 \mathbb{P}^2 's on \mathcal{S}_3 , by Holzapfel. The result is: there are automorphic forms G_2, G_3 and G_4 of indicated weights on \mathbb{B}_2 such that

$$C_\tau = \{y^3 = x^4 - G_2(\tau)x^2 - G_3(\tau)x - G_4(\tau)\}, \quad (48)$$

much akin to the Weierstraß equation for an elliptic curve. (The variable x in (48) is of course different than that in (46)). There is no doubt a similar expression for $\tau \in \mathbb{B}_3$.

3 The Igusa quartic \mathcal{I}_4

This variety has been known since the last century, and it is related to the configuration in \mathbb{P}^4 which is dual to the 15 hyperplanes of (38) and the 15 planes of (31) which they cut out on \mathcal{S}_3 , and in fact \mathcal{I}_4 is just the dual variety of \mathcal{S}_3 . It was also known in the last century that the tangent hyperplane sections of \mathcal{I}_4 are Kummer surfaces, giving \mathcal{I}_4 a moduli interpretation. Igusa, in the 1960's, made this rigorous and showed that \mathcal{I}_4 is the Satake compactification of $\Gamma(2) \backslash \mathbb{S}_2$, the Siegel modular threefold of level 2. We begin by discussing the projective variety, then turn to Igusa's results.

3.1 The quartic locus associated to a configuration of 15 lines

Let l_σ be the line dual in \mathbb{P}^4 to the \mathbb{P}^2 of (31), and let h_{ij} denote the point dual to \mathcal{H}_{ij} of (38). Then these 15 lines meet at the 15 points h_{ij} , and three of the 15 lines meet at each, corresponding to the three \mathbb{P}^2 's which are contained in each \mathcal{H}_{ij} . Furthermore, each of the 15 lines contains three of the 15 points, as each \mathbb{P}^2 is contained in three of the \mathcal{H}_{ij} . It is useful to introduce the following notation: each line is given a notation (ij) , and two

such lines (ij) , (kl) meet if and only if the sets (ij) , (kl) are disjoint. Hence the 15 points are numbered by *synthemes* (ij, kl, mn) and the three lines meeting each point are the indicated *duads* (pairs) (ij) , (kl) , (mn) . Then there are ten sets such as 23, 31, 12 and 56, 64, 45 with the property that the first and last three do not meet, but each of the first meets each of the last. Therefore the six lines are generators of a quadric surface

$$Q_{ijk} = \text{quadric with } (ij), (jk), (ik) \text{ in one ruling and } (lm), (mn), (ln) \text{ in the other ruling} \quad (49)$$

Then Q_{ijk} lies in a \mathbb{P}^3 , and there are ten such, corresponding to the ways of dividing the six numbers into two *triads* (triples). Let us denote the corresponding \mathbb{P}^3 by K_{ijk} , so

$$Q_{ijk} \subset K_{ijk}. \quad (50)$$

Then each of the 15 lines is contained in four of the K_{ijk} , and six of the K_{ijk} meet at each of the 15 points.

Consider now a set of four mutually skew of the 15 lines, for example 12, 23, 24, 25. Then there will be a two-dimensional space of \mathbb{P}^2 's which meet all four lines (as we are in \mathbb{P}^4 , generically a plane and a line will not intersect). Of all of these planes, there are exactly two passing through a given point of space $x \in \mathbb{P}^5$. The locus we are interested in is:

$$\mathcal{Q} := \left\{ x \in \mathbb{P}^5 \left| \begin{array}{l} \text{the two planes meeting four skew lines} \\ \text{of the 15 } (ij) \text{ and passing through } x \\ \text{coincide} \end{array} \right. \right\}. \quad (51)$$

If, as in (34), we take coordinates $\xi, \eta, \zeta, \xi', \eta', \zeta'$ satisfying $\xi + \eta + \zeta = \xi' + \eta' + \zeta'$ as coordinates on \mathbb{P}^4 , then the condition (51) yields a locus with equation ([B], p. 125):

$$\sqrt{(\eta - \zeta')(\eta' - \zeta)} + \sqrt{(\zeta - \xi')(\zeta' - \xi)} + \sqrt{(\xi - \eta')(\xi' - \eta)} = 0. \quad (52)$$

To find the dual variety of the locus \mathcal{Q} , Baker does the following. Letting a, b, c be variables, $a' = (1-a)$, $b' = (1-b)$, $c' = (1-c)$, consider the six points which are the vertices of a coordinate simplex in \mathbb{P}^5 , and call them A, B, C, A', B', C' . Then any point of our \mathbb{P}^4 can be written as $x = A/bc' + B/ca' + C/ab' + A'/b'c + B'/c'a + C'/a'b$. Calculating the tangent plane of \mathcal{Q} at a point $x \in \mathcal{Q}$ which satisfies (52), in terms of the coordinates used in (52), one gets:

$$bc'\xi + ca'\eta + ab'\zeta - b'c\xi' - c'a\eta' - a'b\zeta' = 0. \quad (53)$$

Now putting $u = bc'$, $v = ca'$, $w = ab'$, $u' = -b'c$, $v' = -c'a$, $w' = -a'b$, the equation becomes

$$u\xi + v\eta + w\zeta + u'\xi' + v'\eta' + w'\zeta' = 0, \quad (54)$$

with the two identities

$$u + v + w + u' + v' + w' = 0, \quad uvw + u'v'w' = 0. \quad (55)$$

Since the identities (55) do not depend on the point, it follows that these equations define the dual variety. Now comparing with (34), we have

Proposition 3.1 *The dual variety of the quartic locus \mathcal{Q} is the Segre cubic \mathcal{S}_3 .*

It is easy to see that \mathcal{Q} is singular along the 15 lines. It was also noted classically that a tangent hyperplane section of \mathcal{Q} is a Kummer quartic surface, with 16 nodes, 15 from the intersections with the 15 singular lines, and one from the point of tangency.

3.2 Igusa's results

The relation to the Kummer quartic surfaces is correctly understood by studying theta constants for the theta functions with $1/2$ -characteristics. This was done by Igusa in [I], and we now recall some of his results.

3.2.1 Theta functions

Let $\tau \in \mathbb{S}_g = \{M \in M_g(\mathbb{C}) \mid \tau = {}^t(\tau), \text{Im}(\tau) \text{ positive definite}\}$, $z \in \mathbb{C}^g$, and $m = (m', m'') \in \mathbb{Q}^{2g}$. Note that \mathbb{S}_g is a hermitian symmetric space of type **III_g**.

Definition 3.2 The *theta function of degree g and characteristic m* is defined by the power series

$$\theta_m(\tau, z) = \sum_{n \in \mathbb{Z}^g} \exp\left(\frac{1}{2} {}^t(n + m')\tau(n + m') + {}^t(n + m')(z + m'')\right).$$

As a function of τ the series θ_m converges precisely for $\tau \in \mathbb{S}_g$, while as functions of z by fixed τ these are theta functions on $A_\tau = \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g)$. As such the zeros on A_τ are determined by the characteristic m . The corresponding *theta constant* is

$$\theta_m(\tau) := \theta_m(\tau, 0). \quad (56)$$

Igusa has studied in [I] these theta constants, in particular the theta functions with characteristics $m \in \frac{1}{2^n}\mathbb{Z}$. Some of his results are the following.

Lemma 3.3 $\theta_m(\tau) \equiv 0 \iff m \bmod(1)$ satisfies $\exp(4\pi i({}^t m')m'') = -1$.

The Siegel modular group $\Gamma_g(1) = Sp(2g, \mathbb{Z})$ acts on the arguments (τ, z) as follows:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad M(\tau, z) = ((A\tau + B)(C\tau + D)^{-1}, (C\tau + D)^{-1}z), \quad (57)$$

and on the characteristic itself by

$$M(m) = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \cdot m + \frac{1}{2} \begin{pmatrix} \text{diag}(C^t D) \\ \text{diag}(A^t B) \end{pmatrix}. \quad (58)$$

The behavior of the theta functions under M is given by

Lemma 3.4 Let $M \in \Gamma_g(1)$ act on (τ, z) as in (57) and on the characteristic m as in (58). Then the theta functions transform according to the rule:

$$\theta_{M(m)}(M(\tau, z)) = \kappa(M) \exp(2\pi i \phi_m(M)) \det(C\tau + D)^{1/2} \times \exp(\pi i {}^t z (C\tau + D)^{-1} C z) \theta_m(\tau, z), \quad (59)$$

where $\kappa(M)$ is some eighth root of unity and $\phi_m(M)$ is defined by the formula

$$\phi_m(M) = -\frac{1}{2} {}^t m' B D m' + {}^t m'' A C m'' - 2 {}^t m' A C m'' - {}^t \text{diag}(A^t B) (D m' - C m'').$$

In particular for the theta constants the formula becomes

$$\theta_{M(m)}(M\tau) = \kappa(M) \exp(2\pi i \phi_m(M)) \det(C\tau + D)^{1/2} \theta_m(\tau). \quad (60)$$

What the equation (60) says for $g = 2$ is that up to an eighth root of unity, non-vanishing theta constants with $1/2$ -characteristics are automorphic forms of weight $1/2$ for the main congruence subgroup of level 2 in $Sp(4, \mathbb{Z})$. Indeed, for $M \in \Gamma(2)$, it holds that $e^{2\pi i \phi_m(M)} = 1$, as Igusa shows. There are 16 characteristics m ; six are *odd* (i.e., $\theta_m(\tau, z) = -\theta_m(\tau, -z)$) so give rise to vanishing theta constants, while ten are even. The fourth powers θ_m^4 are genuine automorphic forms for $\Gamma(2)$, and determine a morphism

$$f : \Gamma(2) \backslash \mathbb{S}_2 \longrightarrow \mathbb{P}^9 = (\theta_{m_1}^4 : \dots : \theta_{m_{10}}^4), \quad (61)$$

where m_1, \dots, m_{10} are the ten even characteristics.

3.2.2 The ring of automorphic forms

Among the ten coordinate theta functions there are five linear relations, the Riemann relations. This implies that the map f in (61) maps into a \mathbb{P}^4 , displaying the quotient $X_{\Gamma(2)}$ as a hypersurface. In fact, since this is an embedding by means of automorphic functions whose closure $X_{\Gamma(2)}^* \subset \mathbb{P}^4$ is normal (see below), it follows that f gives a Baily-Borel embedding of the arithmetic quotient. The proof that f is an embedding given by Igusa is quite deep, involving showing that the ring of modular forms of $\Gamma(2)$ is the integral closure of the ring generated by the said theta functions. More precisely, his result is

Theorem 3.5 ([I], p. 397) *Take as coordinates in \mathbb{P}^4 the following theta constants:*

$$\begin{aligned} y_0 &= \theta_{(0110)}^4(\tau), \quad y_1 = \theta_{(0100)}^4(\tau), \quad y_2 = \theta_{(0000)}^4(\tau), \\ y_3 &= \theta_{(1000)}^4(\tau) - \theta_{(0000)}^4(\tau), \quad y_4 = -\theta_{(1100)}^4(\tau) - \theta_{(0000)}^4(\tau), \end{aligned}$$

where we let $(ijkl)$ denote the characteristic $(\frac{i}{2} \frac{j}{2} \frac{k}{2} \frac{l}{2})$. Set also

$$\chi_{10} = \prod_{\text{even } m} \theta_m^2.$$

Then the ring of modular forms of $\Gamma(2)$ is given by:

$$R(\Gamma(2)) = \mathbb{C}[y_0, \dots, y_4, \chi_{10}]/\mathcal{E},$$

where \mathcal{E} is the ideal generated by the following two relations:

$$\mathcal{E} = \begin{cases} R_1 &= (y_0 y_1 + y_0 y_2 + y_1 y_2 - y_3 y_4)^2 - 4 y_0 y_1 y_2 (\sum y_i) \\ R_2 &= \chi_{10}^2 - \frac{1}{4} s(y_0, \dots, y_4), \quad s \text{ homogenous of degree } 5 \end{cases}$$

However, the formula R_1 relating the theta functions was known long before Igusa. Since the five linear relations determining the image \mathbb{P}^4 of f are known, it is sufficient to give a single relation of minimal degree among the θ_m^4 to determine the image. This relation can be found as early as in the 1887 paper of Maschke [Ma], p. 505². In terms of the theta constants above, this equation is

$$\left(\sum \theta_m^8 \right)^2 - 4 \left(\sum \theta_m^{16} \right) = 0, \quad (62)$$

which, as can be checked, is the same quartic as that given by R_1 in 3.5, as well as that given by (52).

Definition 3.6 The *Igusa quartic* \mathcal{I}_4 is the quartic threefold defined in \mathbb{P}^4 by the relation R_1 of Theorem 3.5 or the equation (62).

As a corollary we have

Corollary 3.7 *The Igusa quartic \mathcal{I}_4 and the quartic locus \mathcal{Q} of (52) coincide, and this quartic is the Satake compactification of $X_{\Gamma(2)}$.*

Hence we have described $X_{\Gamma(2)}^*$ as a singular quartic hypersurface in \mathbb{P}^4 . There are the two interesting loci:

- (i) the singular locus, which is the boundary of the Baily-Borel embedding of $X_{\Gamma(2)}$;
- (ii) the intersection of \mathcal{I}_4 with the coordinate hyperplanes in \mathbb{P}^9 , which are the modular subvarieties $\overline{Y}_m(2)$ of [J], Thm. 3.19; these are quotients of symmetric subdomains isomorphic to a product of discs.

As already mentioned, the singular locus of \mathcal{I}_4 consists of 15 lines; this can be directly calculated from the equation. Alternatively, applying general formula for the number of cusps (see for example [Y]) we see that $X_{\Gamma(2)}$ has 15 one-dimensional boundary components and 15 zero-dimensional boundary components; by 3.7 this is then the singular locus of \mathcal{I}_4 . (That these boundary components are rational curves is obvious ($\Gamma(2) \backslash \mathbb{S}_1$ is rational); that they are actually *lines* is not so obvious, but an easy calculation). This line of reasoning also requires the result, also due to Igusa, that, although $\Gamma(2)$ is not torsion-free, there are nonetheless no singularities on $X_{\Gamma(2)}$.

²the equation is somewhat hidden: "...daß dagegen die symmetrische Function vierter Dimension sich bis auf einen Zahlenfactor als das Quadrat der zweiten Dimension erweist."

3.3 Moduli interpretation

The embedding (61) of $X_{\Gamma(2)}^*$ as the quartic \mathcal{I}_4 shows that \mathcal{I}_4 has a moduli interpretation. In fact, $X_{\Gamma(2)}$ is a rough moduli space of principally polarised abelian surfaces with a level 2 structure. However, $\Gamma(2)$ contains torsion, namely the element -1 , so $X_{\Gamma(2)}$ is *not* a fine moduli variety. This corresponds to the fact that the automorphism $z \mapsto -z$ of A_τ *preserves* the level 2 structure, hence the actual *object* which is parameterised by $X_{\Gamma(2)}$ is the *quotient* $A_\tau/(z \mapsto -z)$. This is just the Kummer quartic surface which already occurred above. The precise relation is given by

Theorem 3.8 *For a point $x \in \mathcal{I}_4 - \{\text{intersections of } \mathcal{I}_4 \text{ with the ten coordinate planes in (61)}\}$, the corresponding Kummer quartic surface $K_x = A_\tau/\{\pm 1\}$, where $x = p(\tau)$ for the natural projection $p : \mathbb{S}_2 \rightarrow X_{\Gamma(2)}$, is the intersection of \mathcal{I}_4 with the tangent hyperplane at x , $T_x\mathcal{I}_4$:*

$$K_x = \mathcal{I}_4 \cap T_x\mathcal{I}_4.$$

This statement can be found for example in [B]. It amounts to the fact, true in any dimension, that for $n \geq 3$ the theta functions with characteristics $\in \mathbb{Z}/n\mathbb{Z}$ on a fixed A_τ give an embedding of A_τ , while for $n = 2$ they map onto the Kummer variety.

The reason one must exclude the ten hyperplane sections in Theorem 3.8 is the following result.

Proposition 3.9 *The ten hyperplane sections $\{\theta_m^4 = 0\} \cap \mathcal{I}_4$ are tangent hyperplane sections, i.e., the intersection is of degree 2 and multiplicity 2.*

A proof, based only on the equation of \mathcal{I}_4 , can be found in [B]. To understand the meaning of this, note that a general hyperplane section meets \mathcal{I}_4 in a quartic surface, while the intersections here are quadric surfaces, hence to preserve degree must be counted twice (i.e., multiplicity 2). Consider the symmetric subdomain $\mathbb{S}_1 \times \mathbb{S}_1 \subset \mathbb{S}_2$, which in this case is the set of reducible matrices:

$$\mathbb{S}_1 \times \mathbb{S}_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} \right\} = \mathbb{S}_2.$$

Then an easy calculation shows that the theta function of Definition 3.2 is a *product* of two theta functions of a single variable (i.e., $z \in \mathbb{C}$). This is equivalent to the fact that for reducible $\tau \in \mathbb{S}_2$, the abelian surface A_τ is a product of two elliptic curves, $A_\tau = E_1 \times E_2$. In this case, the map given onto the ‘‘product Kummer’’ variety is a map $s : E_1 \times E_2 \rightarrow E_1/\{\pm 1\} \times E_2/\{\pm 1\} = \mathbb{P}^1 \times \mathbb{P}^1$, and this $\mathbb{P}^1 \times \mathbb{P}^1$ is the quadric surface occurring in 3.9. Since $\mathbb{P}^1 \times \mathbb{P}^1$ has no moduli, we see that *formally* the statement of Theorem 3.8 remains true for all $x \in \mathcal{I}_4 - \{15 \text{ singular lines}\}$, if we consider product Kummer varieties instead of the usual ones, and the hyperplane section is the quadric surface of Proposition 3.9. Note however, that this quadric surface, being a modular subvariety, can also be described as:

$$E_1/\{\pm 1\} \times E_2/\{\pm 1\} \cong (\Gamma_1(2)\backslash\mathbb{S}_1)^* \times (\Gamma_1(2)\backslash\mathbb{S}_1)^*, \quad (63)$$

describing the product Kummer surface of a reducible abelian surface as a compactification of an arithmetic quotient, that is, as a Janus-like variety. We then get the following moduli interpretation of the quadric surfaces.

Proposition 3.10 *The ten quadric surfaces of Proposition 3.9 are modular subvarieties which correspond to abelian surfaces which split. More precisely, for any x on one of the quadric surfaces, but not on any of the singular lines (there are six such singular lines on each quadric surface, see (49)), determines a smooth abelian surface which splits, with a level 2 structure.*

Finally we note that this geometry can be described, as discussed already in [J] and many other places, in terms of the finite geometry of

$$V = (\mathbb{Z}/2\mathbb{Z})^4. \quad (64)$$

Let \langle, \rangle denote the induced symplectic form on V ; every vector $v \in V$ is isotropic with respect to \langle, \rangle . Since there are 15 non-zero vectors, there are 15 one-dimensional boundary components. Similarly, there are 15 isotropic planes in V , giving 15 zero-dimensional boundary components. The modular subvarieties of Proposition 3.10 correspond in this setting to non-singular pairs $\{\delta, \delta^\perp\}$, where δ is a two-dimensional subspace of V on which \langle, \rangle is non degenerate, and δ^\perp denotes the orthocomplement with respect to \langle, \rangle . Of these there are exactly ten, as is easily checked. We leave further details to the reader.

3.4 Birational transformations

We have seen above in Proposition 3.1 that \mathcal{S}_3 and \mathcal{I}_4 are dual varieties. It follows from general theory that they are then in fact *birational*. In this section we describe the ensuing birational map explicitly. We consider the following modifications of \mathbb{P}^4 .

- a) Blow up the ten nodes (30) of \mathcal{S}_3 ; denote this by $\varrho_1 : \widehat{\mathbb{P}}^4 \longrightarrow \mathbb{P}^4$. There are ten exceptional \mathbb{P}^3 's, each with normal bundle $\mathcal{O}_{\mathbb{P}^3}(-1)$. Consider one of the 15 hyperplanes \mathcal{H}_{ij} of (38). Since each hyperplane contains $4 + 2 + 1 = 7$ nodes, its proper transform on $\widehat{\mathbb{P}}^4$ is a \mathbb{P}^3 blown up in those seven points; each of the 15 \mathbb{P}^2 's of (31) lying on \mathcal{S}_3 contains four of the nodes, so their proper transforms are copies of \mathbb{P}^2 blown up in four points. Finally, let $\widehat{\mathcal{S}}_3 \subset \widehat{\mathbb{P}}^4$ denote the proper transform of \mathcal{S}_3 in $\widehat{\mathbb{P}}^4$; $\widehat{\mathcal{S}}_3$ is smooth, and $\varrho_1|_{\widehat{\mathcal{S}}_3} : \widehat{\mathcal{S}}_3 \longrightarrow \mathcal{S}_3$ is a desingularisation of \mathcal{S}_3 , replacing each node with a quadric surface $\cong \mathbb{P}^1 \times \mathbb{P}^1$.
- b) Let $\mathcal{J}(15)$ denote the ideal of the 15 singular lines of \mathcal{I}_4 ; blow up $\mathcal{J}(15)$, and let $\varrho_2 : \widehat{\mathbb{P}}^4 \longrightarrow \mathbb{P}^4$ denote this modification. Under ϱ_2 , each of the lines is replaced by a \mathbb{P}^2 -bundle over that line, and each point is replaced by a union of \mathbb{P}^1 's, one each for each *pair* (l_1, l_2) of *lines* meeting at the point; this mentioned \mathbb{P}^1 is then the intersection of the fibre \mathbb{P}^2 of ϱ_2 at that point with the (two) exceptional \mathbb{P}^2 -bundles over the lines l_1 and l_2 . Note that the proper transforms of the ten quadrics of Proposition 3.9 on \mathcal{I}_4 are still biregular to $\mathbb{P}^1 \times \mathbb{P}^1$, while the proper transforms of each of the lines turns out to be a *Kummer modular surface*, that is, \mathbb{P}^2 blown up in four points. Let $\widehat{\mathcal{I}}_4$ be the proper transform of \mathcal{I}_4 in $\widehat{\mathbb{P}}^4$; then $\varrho_2|_{\widehat{\mathcal{I}}_4} : \widehat{\mathcal{I}}_4 \longrightarrow \mathcal{I}_4$ is a desingularisation of \mathcal{I}_4 .

Theorem 3.11 *The varieties $\widehat{\mathcal{S}}_3$ and $\widehat{\mathcal{I}}_4$ are biregular, and the explicit birational map $\varphi : \mathcal{S}_3 \dashrightarrow \mathcal{I}_4$ is the birational morphism completing the following diagram:*

$$\begin{array}{ccccc} \widehat{\mathcal{S}}_3 & \xrightarrow{\widetilde{\varphi}} & \widehat{\mathcal{I}}_4 & & \\ \varrho_1 \downarrow & & \downarrow & \varrho_2 & \\ \mathcal{S}_3 & \xrightarrow{\varphi} & \mathcal{I}_4 & & \end{array}$$

Moreover, φ is Σ_6 -equivariant.

Proof: As ϱ_1 and ϱ_2 are Σ_6 -equivariant, the second statement follows from the first. Let $D \subset \mathcal{S}_3$ be the open set:

$$D = \mathcal{S}_3 - \{15 \text{ hyperplanes } P_\sigma \text{ of (31)}\}; \quad (65)$$

here we may take the regular map of D onto the set of tangent hyperplanes (now viewing \mathcal{I}_4 as the projective dual of \mathcal{S}_3), and set

$$\begin{aligned} \varphi|_D : D &\longrightarrow D' \subset \mathcal{I}_4 \\ x &\mapsto (\mathbb{P}^3)_x = \text{tangent hyperplane to } \mathcal{S}_3 \text{ at } x \end{aligned} \quad (66)$$

Lemma 3.11.1 *The subset $D' \subset \mathcal{I}_4$ is: $D' = \mathcal{I}_4 - \{10 \text{ quadric surfaces of Proposition 3.10}\}$.*

Proof: Suppose $x \in D$; then $(\mathbb{P}^3)_x$ meets D in an irreducible cubic (the union of the P_σ are *all* the linear subspaces contained in \mathcal{S}_3 , so outside of this locus $(\mathbb{P}^3)_x \cap \mathcal{S}_3$ cannot have a linear factor, so, being cubic, must be irreducible), while the ten quadric surfaces are the locus of the tangent hyperplanes meeting \mathcal{S}_3 in one of the nodes, all of which are excluded in D . \square

Now we glue D onto the rest of $\widehat{\mathcal{S}}_3$, and D' onto the rest of $\widehat{\mathcal{I}}_4$. The locus $\Lambda_1 = \widehat{\mathcal{S}}_3 - D$ coincides with $\Lambda_2 = \widehat{\mathcal{I}}_4 - D'$, as follows from the descriptions of the rational maps ϱ_1 and ϱ_2 above. Both the Λ_i consist of ten $\mathbb{P}^1 \times \mathbb{P}^1$'s and 15 rational surfaces, each isomorphic to \mathbb{P}^2 blown up in four points. Hence we can complete $\varphi|_D$ to a biregular isomorphism $\varphi : \widehat{\mathcal{S}}_3 \longrightarrow \widehat{\mathcal{I}}_4$, by fixing an isomorphism $\varphi_\Delta : \Lambda_1 \longrightarrow \Lambda_2$, and setting

$$\varphi(x) = \begin{cases} \varphi|_D(x), & \text{if } x \in D \\ \varphi_\Delta(x), & \text{if } x \in \Lambda_1 \end{cases}$$

completing the proof of Theorem 3.11. \square

The following description is more concrete. If x is one of the nodes of \mathcal{S}_3 , there is a quadric cone of tangent (to \mathcal{S}_3) hyperplanes at x ; so closing up φ maps x to the quadric surface over which the above is a cone, i.e., x is blown up. If x is *not* a node, then there is a unique tangent hyperplane $T_x\mathcal{S}_3$, determining a point of \mathcal{I}_4 . Furthermore, $T_x\mathcal{S}_3$ and $T_y\mathcal{S}_3$ coincide for $x \neq y$, if and only if x and y are contained in a common Segre plane (31), and the line joining x and y in that Segre plane passes through one of the four nodes, say N , in that Segre plane. This is because $T_x\mathcal{S}_3 \cap \mathcal{S}_3 = \mathcal{H} \cup Q_x$, where Q_x is a residual quadric cone, and the quadric cone is the intersection of $T_x\mathcal{S}_3$ with the cone C_N which is the tangent cone of the node N in the Segre plane. So if x and y lie on a line through N , Q_x and Q_y coincide, so $T_x\mathcal{S}_3$ and $T_y\mathcal{S}_3$ coincide also.

Theorem 3.12 *The duality map $d : \mathcal{S}_3 \dashrightarrow \mathcal{I}_4$ is given by the linear system of quadrics 2.3, i.e., by the elements of the ideal $\mathcal{J}(10)$ of the ten nodes: $d = \varphi$.*

Proof: It suffices to check that d , viewed as a modification of \mathcal{S}_3 , coincides with the birational map φ of Theorem 3.11. But this is easy. As the base locus is the set of nodes, these are blown up. As just explained, x and y in one of the Segre planes map to the same point on the image line precisely when the line joining them passes through one of the nodes in the Segre plane. As these lines are precisely what the map φ blows down, d certainly coincides with φ . \square

We also have the following analogue of Corollary 2.4.

Lemma 3.13 *The ideal $\mathcal{J}(15)$ of the 15 singular lines of the Igusa quartic coincides with the Jacobian ideal of \mathcal{I}_4 .*

Proof: Once again the inclusion $\mathcal{J}ac(\mathcal{I}_4) \subset \mathcal{J}(15)$ is obvious, and the inverse inclusion can be verified by means of standard basis computations, namely that $\mathcal{J}(15)$ is generated by five cubics. \square

Along the same lines as Theorem 3.12 we then get

Theorem 3.14 *The duality map $d : \mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ is given by the system of cubics containing the 15 lines, i.e., by the Jacobian ideal of \mathcal{I}_4 : $d = \varphi^{-1}$.*

Proof: As above, it suffices to show that d , viewed as a modification of \mathcal{I}_4 , coincides with the map φ^{-1} of Theorem 3.11. This is readily verified, as the base locus, the 15 lines, are blown up, while the tangent planes for any two points x and y in a common quadric of \mathcal{I}_4 (of Proposition 3.9) coincide, blowing down the quadric surface to a node. \square

3.5 The Siegel modular threefold of level 4

From the general theory of congruence subgroups, $X_{\Gamma(4)} \rightarrow X_{\Gamma(2)}$ is a Galois cover, with Galois group $\Gamma(2)/\Gamma(4) \cong (\mathbb{Z}/2\mathbb{Z})^9$. Identifying $X_{\Gamma(2)}^*$ with \mathcal{I}_4 and identifying \mathcal{I}_4 birationally with \mathcal{S}_3 , we can consider Fermat covers over $X_{\Gamma(2)}^*$, i.e., given by a diagram

$$\begin{array}{ccccc} Z(\mathbf{A}_4, n) & \xrightarrow{\overline{\varphi}^{-1}} & Y^\wedge(\mathbf{A}_4, n) & \longleftarrow & \tilde{Y}(\mathbf{A}_4, n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_4 & \xrightarrow{\varphi^{-1}} & \mathcal{S}_3 & \longleftarrow & \tilde{\mathcal{S}}_3 \end{array} \quad (\mathbb{Z}/2\mathbb{Z})^9 \quad (67)$$

where $\overline{\varphi}^{-1}$ is induced by φ^{-1} , that is, (67) is a fibre square (cf. (43), where $\tilde{\mathcal{S}}_3$ is denoted $\tilde{\mathbb{P}}^3$).

Theorem 3.15 *The Fermat cover $Z(\mathbf{A}_4, 2)$ is the Satake compactification of the Siegel modular threefold of level 4.*

Proof: It suffices to show that $\tilde{Y}(\mathbf{A}_4, 2)$ is the induced cover over $\widehat{\mathcal{I}}_4$, where $\varrho_2 : \widehat{\mathcal{I}}_4 \rightarrow \mathcal{I}_4$ is the desingularisation of \mathcal{I}_4 of Theorem 3.11. Now the identification can be reduced to identifying what is in the branch locus of $\tilde{Y}(\mathbf{A}_4, 2) \rightarrow \tilde{\mathcal{S}}_3$. There are two kinds of components:

- a) covers $\tilde{Y}(\mathbf{A}_3, 2)$ of blown up \mathbb{P}^2 's, the H_{ij} of (18);

b) covers $\tilde{Y}(\mathbf{A}_2, 2) \times \tilde{Y}(\mathbf{A}_2, 2)$ of $\mathbb{P}^1 \times \mathbb{P}^1$'s, the L_{0ij} .

Lemma 3.15.1 $\tilde{Y}(\mathbf{A}_3, 2) \cong S(4)$, Shioda's elliptic modular surface of level 4.

Proof: This is well-known. $\tilde{Y}(\mathbf{A}_3, 2)$ is K3 since it is a Fermat cover branched over six lines. One constructs structures of fibre space $\tilde{Y}(\mathbf{A}_3, 2) \rightarrow \mathbb{P}^1$ with elliptic curves as fibres by taking the cover of the pencil of lines through a node (each such line meets four of the six lines outside the node, so the cover is branched at four points, i.e., is elliptic). The six fibres of type I_4 are readily identified, as are the 16 sections. \square

Lemma 3.15.2 The cover $\tilde{Y}(\mathbf{A}_2, 2) \rightarrow \mathbb{P}^1$ coincides with the cover $(\Gamma_1(4) \backslash \mathbb{S}_1)^* \rightarrow \mathbb{P}^1$, by which we mean the Galois actions coincide.

Proof: This is even more well-known. \square

The theorem now follows, provided we accept that $\tilde{Y}(\mathbf{A}_4, 2)$ is a quotient of \mathbb{S}_2 at all, i.e., that the cover $\mathbb{S}_2 \rightarrow \mathcal{I}_4^0$ factorises (here $\mathcal{I}_4^0 = \mathcal{I}_4 - \{15 \text{ lines}\}$), $\tilde{Y}(\mathbf{A}_4, 2)^0 := \tilde{Y}(\mathbf{A}_4, 2) - q^{-1}(15 \text{ lines})$:

$$\begin{array}{ccc} \mathbb{S}_2 & \xrightarrow{\quad} & \mathcal{I}_4^0 \\ & \searrow & \nearrow q \\ & & \tilde{Y}(\mathbf{A}_4, 2)^0. \end{array}$$

(68)

But there is an easy way to see that this is the case: we can, for any given $x \in \tilde{Y}(\mathbf{A}_4, 2) - \Delta$ and $y = q(x)$, put a level 4 structure on A_y , such that the Galois group just permutes the level 4 over level 2 structures, that is, we make the identification $\Gamma(2) \backslash \Gamma(4) \cong (\mathbb{Z}/2\mathbb{Z})^9 \cong$ the Galois group of the cover. So $\tilde{Y}(\mathbf{A}_4, 2)$, being a moduli space as in Shimura's theory, is a quotient of \mathbb{S}_2 . \square

One could also imagine arguing with uniqueness of Galois covers, since we know the branch locus, branch degrees and Galois group. However there is in general no such uniqueness of covers, so we have to be careful. In our situation, there are two possible approaches to show uniqueness:

- 1) Since the modular subvarieties determine, on the group-theoretic side, generators of the corresponding arithmetic group, we could conclude, from the isomorphisms 3.15.1 and 3.15.2, the desired result.
- 2) Since the branch divisors are totally geodesic with respect to the Bergmann metric, on the cover the metric retains its symmetry property.

Method 1) has been applied in [J], and 2) can be carried out for ball quotients.

4 The Hessian varieties of \mathcal{S}_3 and \mathcal{I}_4

4.1 The Nieto quintic

Let $(x_0 : \dots : x_5)$ be the projective coordinates on \mathbb{P}^5 used to define \mathcal{S}_3 in (29), and let $\sigma_i = \sigma_i(x_0, \dots, x_5)$ be the i -th elementary symmetric function $\sigma_\lambda = \sum_{i_1 < \dots < i_\lambda} x_{i_1} \cdots x_{i_\lambda}$ in $(x_0 : \dots : x_5)$. Define the *Nieto quintic* \mathcal{N}_5 by the equations

$$\mathcal{N}_5 = \left\{ \begin{array}{l} \sigma_1 = 0 \\ \sigma_5 = 0 \end{array} \right\} \subset \mathbb{P}^5 = \{\sigma_1 = 0\} \subset \mathbb{P}^5. \quad (69)$$

The symmetry of \mathcal{N}_5 under the symmetric group Σ_6 is evident from the equation. This quintic was discovered in the thesis [N] and further studied in [BN], which will be our general reference for this section. We just briefly describe the geometry of \mathcal{N}_5 without discussing details.

The singular locus is relatively easy to determine, just by calculating the Jacobian of (69). The result is

Proposition 4.1 ([BN], 3.1) \mathcal{N}_5 has the following singular locus:

- (i) 20 lines $L_{ijk} = \{x_i = x_j = x_k = 0 = \sum x_i\}$;
- (ii) ten isolated points, the Σ_6 -orbit of $(1, 1, 1, -1, -1, -1)$, which are the points $P_{ij} = (1, \pm 1, \dots, \pm 1)$, with +1 in the i -th and j -th positions.

We will give a different proof of this below, see the discussion following Proposition 8.2. Note that the ten points occurring in (ii) are just the ten nodes of \mathcal{S}_3 (see (30), cf. also Remark 4.8 below). Furthermore, a local calculation shows that the singularities of \mathcal{N}_5 along the lines of (i) are of the type $\{\text{disc}\} \times A_1$, and at the points of (ii) are ordinary double points. Hence the former are resolved by a \mathbb{P}^1 -bundle over the line L_{ijk} , while the points are resolved, as with the case of \mathcal{S}_3 , by quadric surfaces. The 20 lines L_{ijk} of 4.1 meet at the following 15 points:

$$Q_{ij} = (0, \dots, 1, \dots, -1, \dots) = \{\Sigma_6\text{-orbit of } Q_{56} = (0 : 0 : 0 : 0 : 1 : -1)\}. \quad (70)$$

Lemma 4.2 The 20 lines L_{ijk} of Proposition 4.1 meet four at a time at the 15 points Q_{ij} ; each line L_{ijk} contains three of the points, namely we have $Q_{ij} \in L_{klm} \iff \{i, j\} \cap \{k, l, m\} = \emptyset$.

Proof: The line L_{123} contains the three points Q_{46} , Q_{45} and Q_{56} , so by Σ_6 -invariance each line contains three of the Q_{ij} . The point Q_{56} is contained in the four lines L_{123} , L_{124} , L_{134} and L_{234} , so by Σ_6 -invariance, each point is contained in four lines. \square

Also, \mathcal{N}_5 contains a finite number of linear planes.

Lemma 4.3 \mathcal{N}_5 contains the following 30 \mathbb{P}^2 's:

- (i) 15 planes $N_{ijkl} = \{x_i + x_j = x_k + x_l = x_m + x_n = 0\}$;
- (ii) 15 planes $N_{ij} = \{x_i = x_j = 0 = \sum_{k \neq i, j} x_k\}$;

Proof: It is immediately verified that these planes satisfy the equation (69). \square

Presumably these are in fact *all* the linear planes contained in \mathcal{N}_5 . Note that the N_{ijkl} are just the 15 planes (31) lying on the Segre cubic \mathcal{S}_3 .

Among the 15 planes N_{ijkl} the common intersections were described in the discussion of the planes P_σ on the Segre cubic (see (14)).

Lemma 4.4 Each plane N_{ijkl} contains the following four of the ten points of 4.1, (ii):

$$P_{km}, P_{kn}, P_{lm} \text{ and } P_{ln};$$

it also contains the following three of the 15 points Q_{ij} of (70): Q_{ij} , Q_{kl} and Q_{mn} .

Proof: Consider N_{0123} ; it contains the four nodes $(1 : -1 : 1 : -1 : 1 : -1)$, $(1 : -1 : -1 : 1 : 1 : -1)$, $(1 : -1 : 1 : -1 : -1 : 1)$ and $(1 : -1 : -1 : 1 : -1 : 1)$ which are the points P_{24} , P_{25} , P_{34} and P_{35} , which gives the first statement by Σ_6 -symmetry (there is an asymmetry in the notation, since we may take $i < j, k < l$ in the notation for N_{ijkl} , and since the first coordinate of P_{ij} may be assumed to be +1). Similarly, N_{0123} contains the three points Q_{01} , Q_{23} and Q_{45} , giving the second statement by Σ_6 -symmetry. \square

We now note that these seven points lie in the plane N_{0123} as in Figure 2. This is in fact easily checked. Note that the lines in N_{0123} , i.e., the intersections with the other N_{ijkl} are *not* the lines of Proposition 4.1; those lines have equations such as $x_0 = x_1 + x_2 = x_1 + x_3 = x_4 + x_5$. However, in the 15 planes N_{ij} of 4.3, several of the 20 singular lines L_{ijk} do lie. In fact, we have

Lemma 4.5 Each N_{ij} contains the four lines L_{ijk} , L_{ijl} , L_{ijm} and L_{ijn} . There are three planes passing through L_{ijk} , namely N_{ij} , N_{ik} and N_{jk} . N_{ij} contains none of the nodes of 4.1, (i), but contains six of the points Q_{ij} of (70), namely Q_{kl} , Q_{km} , Q_{kn} , Q_{lm} , Q_{ln} and Q_{mn} . These six points lie three at a time on the L_{ijk} and form in each N_{ij} a configuration as shown in Figure 3. The three light lines are intersection of N_{01} with N_{ijkl} as indicated.

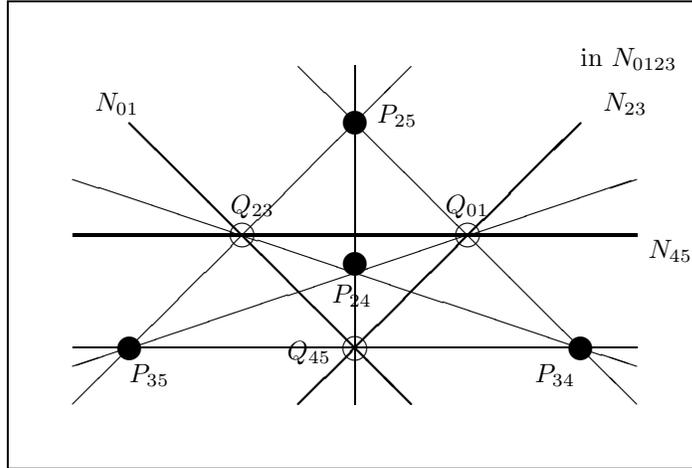


Figure 2: The plane N_{0123}

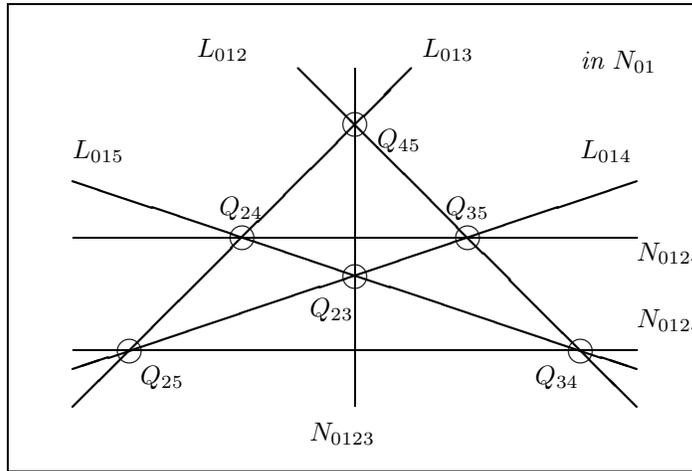


Figure 3: The plane N_{01}

Proof: This is once again easily verified. □

Finally we note that there are hyperplanes in \mathbb{P}^4 cutting out these \mathbb{P}^2 's on \mathcal{N}_5 .

Lemma 4.6 *The six hyperplanes $\mathcal{H}_{ij} = \{x_i + x_j = 0\}$ meet \mathcal{N}_5 each in the union of the three planes N_{ijkl} , N_{ijkm} and N_{ijkn} , and a residual quadric; the six hyperplanes $x_i = 0$ meet \mathcal{N}_5 each in the union of five planes N_{ij} , N_{ik} , N_{il} , N_{im} and N_{in} .*

Proof: This is once again just a computation. □

Now let us consider the intersection of \mathcal{S}_3 and \mathcal{N}_5 . As is obvious from the above description, they both contain the 15 planes N_{ijkl} , and, the intersection being of degree 15, this is the entire intersection. From general arguments on projective varieties, from the fact that the dual of \mathcal{S}_3 , namely the Igusa quartic \mathcal{I}_4 , is *normal*, it follows that the parabolic divisor on \mathcal{S}_3 , which is the intersection of \mathcal{S}_3 with the *Hessian variety*, must get blown down under the duality map, i.e., the intersection $Hess(\mathcal{S}_3) \cap \mathcal{S}_3$ consists of the 15 planes on \mathcal{S}_3 ! Since the Hessian has degree 5, this is the entire intersection, and it is natural to ask whether \mathcal{N}_5 and $Hess(\mathcal{S}_3)$ are related. In fact, we have

Lemma 4.7 *The Nieto quintic is the Hessian of \mathcal{S}_3 , i.e., $\mathcal{N}_5 = Hess(\mathcal{S}_3)$, with equality, not just isomorphism.*

Proof: This is an easy computation (at least for a computer). \square

Remark 4.8 Since the Hessian variety $\text{Hess}(V)$ acquires nodes where V has nodes, this “explains” the ten isolated singularities on \mathcal{N}_5 .

4.2 Two birational transformations

We consider in this section two particularly interesting birational maps from \mathcal{N}_5 .

4.2.1

The first arises through the duality map. Consider the birational map of \mathbb{P}^4 given in the following diagram:

$$\begin{array}{ccccccc} \mathbb{P}^4 & \xleftarrow{\varrho_1} & \widehat{\mathbb{P}}^4 & \xrightarrow{\tilde{\alpha}} & \widehat{\mathbb{P}}^4 & \xrightarrow{\varrho_2} & \mathbb{P}^4 \\ \cup & & \cup & & \cup & & \cup \\ \mathcal{S}_3 & \longleftarrow & \widehat{\mathcal{S}}_3 & \xrightarrow{\sim} & \widehat{\mathcal{I}}_4 & \longrightarrow & \mathcal{I}_4 \end{array} ; \quad (71)$$

ϱ_1 and ϱ_2 were described in section 3.4, and this diagram extends the one of Theorem 3.11 to the ambient rational fourfolds. It is easily seen that the ensuing rational map of \mathbb{P}^4 , $\alpha := \varrho_2 \circ \tilde{\alpha} \circ \varrho_1^{-1} : \mathbb{P}^4 \rightarrow \mathbb{P}^4$, is the map given by the Jacobian ideal of \mathcal{S}_3 , that is, by the linear system of quadrics on the ten nodes of \mathcal{S}_3 (see Corollary 2.4). This “defines” the map $\tilde{\alpha}$; although we could in principle take any extension of $\tilde{\varphi} : \widehat{\mathcal{S}}_3 \rightarrow \widehat{\mathcal{I}}_4$, for our purposes it is convenient to use $\mathcal{J}(10)$. Then set

$$\mathcal{W}_{10} := \alpha(\mathcal{N}_5) \subset \mathbb{P}^4. \quad (72)$$

We now describe \mathcal{W}_{10} and show it is a hypersurface of degree 10, explaining the notation. First we have the

Lemma 4.9 *The map $\alpha : \mathbb{P}^4 \rightarrow \mathbb{P}^4$ blows up the ten nodes P_{ij} of \mathcal{S}_3 , with exceptional divisors \mathcal{E}_{ij} . Let \mathcal{C}_{ij} denote the tangent cone at the point P_{ij} , a quadric cone fibred in lines passing through P_{ij} . Then each line of the cone gets blown down to the corresponding point in \mathcal{E}_{ij} .*

Proof: Since all the quadrics of $\mathcal{J}(10)$ vanish at the P_{ij} , these points are blown up. To say the lines of \mathcal{C}_{ij} get blown down is to say the ratios of the quadrics are constant along the line. This follows from the fact that the quadrics are the partial derivatives of f (the defining polynomial of \mathcal{S}_3), and the line is tangent to the zero locus of f . \square

From this we get

Lemma 4.10 *α , restricted to \mathcal{N}_5 , is an isomorphism on the complement in $\varrho_1^{-1}(\mathcal{N}_5)$ of the intersection of \mathcal{N}_5 with the tangent cones at the ten isolated singularities P_{ij} .*

Proof: This is clear from construction, taking into account the following fact, proved in [BN]: the intersection of \mathcal{N}_5 with the tangent cone of one of the nodes P_{ij} consists of the six Segre planes through the node, and an irreducible quartic ruled surface. It is then clear that these ruled surfaces get blown down, and that outside the Segre cubic and the ruled quartics, the birational map is a morphism. \square

From this it follows:

Lemma 4.11 *The birational map $\varrho_2 \circ \tilde{\alpha} \circ \varrho_1^{-1} : \mathcal{N}_5 \dashrightarrow \mathcal{W}_{10}$ has image \mathcal{W}_{10} , whose singular locus contains the following.*

- (i) ten singular quadric surfaces (the tangent hyperplane intersections of \mathcal{I}_4);
- (ii) 20 singular lines, coming from the singular locus of \mathcal{N}_5 .

Proof: This follows from the description above. \square

Now note that both $Hess(\mathcal{I}_4)$ and \mathcal{W}_{10} are symmetric under Σ_6 , and both are singular along the ten quadrics of the Igusa \mathcal{I}_4 . It may very well be that the two coincide, but we have not checked this. At any rate, \mathcal{W}_{10} meets \mathcal{I}_4 in the union of quadric surfaces, each with multiplicity 2. Hence

Theorem 4.12 $\mathcal{I}_4 \cap \mathcal{W}_{10}$ consists of the ten quadric surfaces (49), each with multiplicity 2. Consequently, the degree of \mathcal{W}_{10} is 10, justifying the notation.

Proof: The intersection has reduced degree 20, and each surface component is counted twice, hence the degree of the intersection is 40, so the degree of \mathcal{W}_{10} is 10. \square

Problem 4.13 Is \mathcal{W}_{10} also a compactification of an arithmetic quotient?

4.2.2

The other birational transformation is the following.

- a) Blow up the 15 points Q_{ij} of (70); let $p_1 : \tilde{\mathcal{N}}_5 \rightarrow \mathcal{N}_5$ denote this blow up.
- b) As each of the lines L_{ijk} contains three points (see Lemma 4.2), each L_{ijk} can be blown down to an isolated singular point (the normal bundle is $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$, cf. (12)).
Let $p_2 : \tilde{\mathcal{N}}_5 \rightarrow \hat{\mathcal{N}}_5$ denote this blow down.

The following is easy to see (see Figures 2 and 3).

Lemma 4.14 *The singular locus of $\hat{\mathcal{N}}_5$ consists of the 20 isolated cusps from (73) b), and the ten cusps, the images of the singular points P_{ij} of Proposition 4.1, (ii). The proper transforms of the N_{ijkl} of Lemma 4.3, (i) on \mathcal{N}_5 are \mathbb{P}^2 's blown up in three points, a del Pezzo surface; the proper transforms of the N_{ij} of Lemma 4.3, (ii) are \mathbb{P}^2 's blown up in six points, then the L_{ijk} are blown down to four nodes, so this is the singular cubic surface with four nodes, the Cayley cubic.*

4.3 Moduli interpretation

The Nieto quintic was discovered as the solution of a certain moduli problem, and we briefly state the results of [BN] describing this.

The point of departure is the action of the Heisenberg group $H_{2,2}$ on \mathbb{P}^3 , and the study of quartics which are invariant under the action. $H_{2,2}$ is a group of order 32 generated by the following linear transformations of \mathbb{P}^3 with coordinates $(z_0 : z_1 : z_2 : z_3)$:

$$\begin{aligned}
 \sigma_1 & : (z_0 : z_1 : z_2 : z_3) \mapsto (z_2 : z_3 : z_0 : z_1) \\
 \sigma_2 & : (z_0 : z_1 : z_2 : z_3) \mapsto (z_1 : z_0 : z_3 : z_2) \\
 \tau_1 & : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : z_1 : -z_2 : -z_3) \\
 \tau_2 & : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : -z_1 : z_2 : -z_3)
 \end{aligned} \tag{74}$$

The center of the group is ± 1 and $PH_{2,2} = H_{2,2}/\pm 1$ has a nice interpretation:

$$PH_{2,2} \cong (\mathbb{Z}/2\mathbb{Z})^4, \tag{75}$$

which carries, as in (64), an induced symplectic form. This means that one can speak of isotropic elements of the group $PH_{2,2}$. The normaliser of $H_{2,2}$ in $SL(4, \mathbb{C})$ maps surjectively to $\Sigma_6 \cong Sp(4, \mathbb{Z}/2\mathbb{Z})$, which acts transitively on diverse geometric loci of the symplectic form *inside* the group $PH_{2,2}$. These loci are:

- a) 15 pairs of skew lines
- b) 15 invariant tetrahedra
- c) ten fundamental quadrics.

Table 3: Singular Heisenberg invariant quartics

$z \in \mathbb{P}^3$	$\dim Q^{sing}$	$Q_{(A,B,C,D,E)}$	$S_{(A,B,C,D,E)}$
\notin fix line	0	Kummer surface	Segre cubic
\in one fix line	2	singular in four coordinate vertices	$A = 0$
\in the intersection of two fixed lines	3	singular along two fixed lines	$A = B = 0$

Notations: Q^{sing} denotes the space of quartics singular at z , $S_{(A,B,C,D,E)}$ denotes the equation of the locus Q^{sing} in the coordinates (A, B, C, D, E) .

The moduli problem considered is a special set of quartics which are invariant under (75). The set of *all* invariant quartics is just a \mathbb{P}^4 , spanned for example by the five quartics:

$$\begin{aligned}
g_0 &:= z_0^4 + z_1^4 + z_2^4 + z_3^4 \\
g_1 &:= 2(z_0^2 z_1^2 + z_2^2 z_3^2) & g_2 &:= 2(z_0^2 z_2^2 + z_1^2 z_3^2) & g_3 &:= 2(z_0^2 z_3^2 + z_1^2 z_2^2) \\
g_4 &:= 4z_0 z_1 z_2 z_3.
\end{aligned}$$

Let (A, B, C, D, E) denote the coordinates of a particular quartic $Q_{(A,B,C,D,E)} = \{Ag_0 + Bg_1 + Cg_2 + Dg_3 + Eg_4 = 0\}$. The generic quartic $Q_{(A,B,C,D,E)}$ is smooth, and the locus of singular quartics can be determined as an equation in (A, \dots, E) . Note that the (A, \dots, E) are functions of $(z_0 : \dots : z_3)$, so the answer as to whether $Q_{(A,B,C,D,E)}$ is singular depends on the point $z \in \mathbb{P}^3$. This is discussed in detail in [BN]. The result is given in Table 3. As one sees, the first row of the table is equivalent to Theorem 3.8 above! The special class of quartics to be considered here is, however, a quite different set, consisting of generically smooth quartics. This is the set of Kummer surfaces of (1,3)-polarised abelian surfaces, which, as it turns out, can be smoothly embedded in \mathbb{P}^3 . This was discovered independently by Naruki and Nieto (see [Na] and [N]). The 16 exceptional \mathbb{P}^1 's resolving the 16 double points of the Kummer surface are 16 disjoint *lines* on the quartics. Also, by a result of Nikulin [Ni], the converse is true, i.e., any quartic containing 16 lines is a Kummer surface. Furthermore, the quartic being invariant under $PH_{2,2}$, if it contains one line, it contains all 16 transforms, so the moduli involved is the condition:

$$L \text{ is a line in } \mathbb{P}^3 \text{ lying on a smooth Heisenberg invariant quartic surface} \quad (77)$$

The equation describing this in the Grassmannian $\mathbb{G}(2, 2) = \{x_0^2 + \dots + x_5^2 = 0\}$ is calculated in [N]. It is

$$\mathcal{M}_{20} = \{\sigma_5(x_0^2, \dots, x_5^2) = 0 = \sigma_1(x_0^2, \dots, x_5^2)\}. \quad (78)$$

Now one considers the natural 2-power map

$$\begin{aligned}
m_2 : \mathbb{P}^5 &\longrightarrow \mathbb{P}^5 \\
(x_0, \dots, x_5) &\mapsto (x_0^2, \dots, x_5^2) = (u_0, \dots, u_5)
\end{aligned} \quad (79)$$

and the image of \mathcal{M}_{20} in \mathbb{P}^5 . Comparing the equations (69) and (78) we have

Lemma 4.15 $m_2(\mathcal{M}_{20}) = \mathcal{N}_5$.

The main results of [BN] can be described as follows. First we define a Zariski open subset $M^s \subset \mathcal{M}_{20}$. The following 15 quadric surfaces $q_{ij} \subset \mathbb{P}^5$ actually lie on \mathcal{M}_{20} , as is easily verified:

$$q_{ij} = \{x_i = x_j = 0 = \sum_{m \neq i,j} x_m^2\}. \quad (80)$$

Under the squaring map m_2 (79) the quadric q_{ij} maps to the plane

$$N_{ij} = \{u_i = u_j = 0 = \sum_{m \neq i,j} u_m\}; \quad (81)$$

so the image of $\mathbf{Q} := \cup_{i,j} q_{ij}$ is $\mathbf{N} := \cup_{i,j} N_{ij}$, and the planes N_{ij} are the 15 planes of Lemma 4.3 (ii). Furthermore the N_{ij} are contained in the branch locus of $m_2|_{\mathcal{M}_{20}} : \mathcal{M}_{20} \rightarrow \mathcal{N}_5$; this locus is *singular* on \mathcal{M}_{20} because \mathcal{N}_5 is tangent to $u_i = 0$ and $u_j = 0$ in all of N_{ij} .

Next consider the inverse image under m_2 of the ten nodes; since the nodes lie on *none* of the branch planes $u_i = 0$, each node has $\deg(m_2) = 32$ inverse images, so \mathcal{M}_{20} has 320 singular points (clearly also nodes), which are the Σ_6 -orbit, call it \mathbf{P} , of the points

$$(\pm 1 : \pm 1 : \pm 1 : \pm i : \pm i : \pm i). \quad (82)$$

Finally consider the inverse images of the 15 Segre planes of Lemma 4.3 (i). This locus is given by the 15 equations which are the Σ_6 -orbit of

$$x_0^2 + x_1^2 = x_2^2 + x_3^2 = x_4^2 + x_5^2 = 0. \quad (83)$$

Inspection shows that this degree 8 surface on \mathcal{M}_{20} splits into eight planes, giving altogether 120=15.8 planes on \mathcal{M}_{20} ; let \mathbf{R} denote their union. Now define:

$$M^s := \mathcal{M}_{20} - \mathbf{Q} - \mathbf{P} - \mathbf{N}, \quad \mathcal{N}_5^s := m_2(M^s). \quad (84)$$

Then the statement proved in [BN] is

Theorem 4.16 *a) M^s is isomorphic to a Zariski open subset of the moduli space $\mathcal{A}_{(1,3)}(2)$ of abelian surfaces with a (1, 3) polarisation and a level 2 structure;*

b) There is a double cover $p : \tilde{\mathcal{N}} \rightarrow \mathcal{N}_5$ for which $p^{-1}(\mathcal{N}_5^s)$ is isomorphic to a Zariski open set of the moduli space $\mathcal{A}_{(2,6)}(2)$;

c) \mathcal{N}_5^s is the moduli space of $PH_{2,2}$ -invariant smooth quartic surfaces containing 16 skew lines.

Since the varieties \mathcal{M}_{20} , $\tilde{\mathcal{N}}$ and \mathcal{N}_5 are compactifications of the Zariski open sets of (84), we have the following:

Corollary 4.17 *There are birational equivalences:*

$$\mathcal{M}_{20} \dashrightarrow (\Gamma_{(1,3)}(2) \backslash \mathbb{S}_2)^*, \quad \tilde{\mathcal{N}} \dashrightarrow (\Gamma_{(2,6)}(2) \backslash \mathbb{S}_2)^*, \quad \mathcal{N}_5 \dashrightarrow (\Gamma \backslash \mathbb{S}_2)^*,$$

where $\Gamma_{(1,3)} \subset \Gamma_{(2,6)} \subset \Gamma$, $[\Gamma : \Gamma_{(2,6)}(2)] = 2$.

As is shown in [BN], the map $\tilde{\mathcal{N}} \rightarrow \mathcal{N}_5$ is given in the following way. It just happens to turn out the *any* of the $PH_{2,2}$ -invariant quartics with 16 skew lines actually contains 32 lines, the first skew set of 16 and a second set of 16 skew lines. The second set of sixteen is found as the image of the first set under the involution

$$(x_0 : \dots : x_5) \mapsto \left(\frac{-1}{x_0} : \frac{1}{x_1} : \dots : \frac{1}{x_5} \right), \quad (85)$$

which can be adjoined to the group $PH_{2,2}$ to form a group of order 32. Altogether the 32 lines have the following properties.

- a) The 32 lines intersect in 32 points;
- b) Each line contains ten of the 32 intersection points; (86)
- c) Each intersection point is contained in ten of the 32 lines.

A configuration with the properties (86) is called a (32_{10}) -configuration.

From Nikulin's results just mentioned, it follows that the second set of 16 lines are also the images of blown-up torsion points on another abelian surface, so there are *two* abelian surfaces with (2, 6) polarisation and level 2 structure giving rise to the *same* resolved Kummer surface, i.e., the map is given by

$$\begin{aligned} \tilde{\mathcal{N}} &\longrightarrow \mathcal{N}_5 \\ (A_{\tau_1}, A_{\tau_2}) &\mapsto \overline{(A_{\tau_1}/\{\pm 1\})} \cong \overline{(A_{\tau_2}/\{\pm 1\})}, \end{aligned} \quad (87)$$

where the isomorphism permutes the two sets of 16 skew lines.

The next step is to identify the modular subvarieties on the arithmetic quotients of Corollary 4.17. From the structure of the periods we know that in terms of abelian surfaces, these modular subvarieties parameterise the abelian surfaces which split. These loci are described to some extent in [BN].

Theorem 4.18 *a) Points on \mathcal{N}_5 parameterise smooth quartic surfaces unless they lie on one of the 30 planes of Lemma 4.3;*

b) points on \mathcal{N}_5 parameterise quartic surfaces containing more than 32 lines if and only if the corresponding abelian surfaces are products. Furthermore, a line on a surface of this set of quartic surfaces has coordinates in \mathbb{P}^5 which is in the Σ_6 -orbit of

$$x_0^4(x_1^2 + x_2^2) + x_1^4(x_2^2 + x_0^2) + x_2^4(x_0^2 + x_1^2) - 6x_0^2x_1^2x_2^2 = 0.$$

Unfortunately, these results do not allow us to explicitly describe the relation between the compactification \mathcal{M}_{20} and compactifications of $\mathcal{A}_{(1,3)}(2)$, in particular the Baily-Borel embedding. This must be considered an interesting open problem.

4.4 A conjecture

To end this section we make a conjecture on one of the birational models of the variety \mathcal{N}_5 . Consider the birational map $\mathcal{N}_5 - - \rightarrow \widehat{\mathcal{N}}_5$ of (73). Recalling now the Janus-like isomorphism between the Picard modular variety $\overline{X}_{\Gamma_{\sqrt{-3}}(\sqrt{-3})}$ and the Siegel modular variety $\overline{X}_{\Gamma(2)}$ (see [J]), it is natural to ask about an analogue here, since the involved Siegel modular varieties of Corollary 4.17 all are related to level 2, albeit with different polarisations. So consider abelian fourfolds with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ of signature (3,1), with a level $\sqrt{-3}$ structure, but with (1,1,1,3) polarisations.

Problem 4.19 Is $\widehat{\mathcal{N}}_5$ the Satake compactification of $X_{(1,1,1,3)}(\sqrt{-3}) := \Gamma_{(1,1,1,3)}(\sqrt{-3}) \backslash \mathbb{B}_3$, where $\Gamma_{(1,1,1,3)}(\sqrt{-3})$ denotes the arithmetic group giving equivalence of complex multiplication by $\mathbb{Q}(\sqrt{-3})$, signature (3,1), with a level $\sqrt{-3}$ structure and a (1,1,1,3)-polarisation?

I conjecture that for *some* subgroup of $\Gamma_{(1,1,1,3)}(\sqrt{-3})$, this does in fact hold.

Evidence for the conjecture:

i) The proper transforms of the 15 Segre planes are by Proposition 2.7 the moduli space of principally polarised abelian threefolds with complex multiplication by $\mathbb{Q}(\sqrt{-3})$, signature (2,1), with a level $\sqrt{-3}$ structure, (although these moduli spaces are blown up in three points on $\widehat{\mathcal{N}}_5$). These could parameterise abelian fourfolds with said CM, signature (3,1) with a level $\sqrt{-3}$ structure and polarisation (1,1,1,3) which split:

$$A_4 \cong A_3 \times A_1,$$

where A_3 has CM, signature (2,1), polarisation (1,1,1), and A_1 has CM, but a polarisation 3.

ii) The proper transforms of the 15 planes N_{ij} of Lemma 4.3, (ii), are four nodal cubic surfaces (Lemma 4.14). These surfaces occur also on the ball quotient \mathcal{S}_3 above: pick any four of the nodes which are not coplanar; they determine a unique \mathbb{P}^3 in \mathbb{P}^4 , and its intersection with \mathcal{S}_3 , a cubic surface, has four nodes in the four nodes of \mathcal{S}_3 in that \mathbb{P}^3 . (Note that there is a unique four-nodal cubic surface, as it is \mathbb{P}^2 blown up in the six intersection points of four (general) lines, a complete quadrilateral in \mathbb{P}^2 , and any two such quadrilaterals are projectively equivalent. This cubic surface is usually called the Cayley cubic, mentioned above.)

iii) The singular locus consists of isolated singular points, resolved by quadric surfaces, so these singularities are rational. Recall that at each P_{ij} , six of the 15 Segre planes meet. At each Q_{ij} (the 15 points (70)), three of the Segre planes and six of the N_{ij} of Lemma 4.3, (ii) meet. In both cases, the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ can be covered equivariantly by a product $E_\rho \times E_\rho$ of the elliptic curve E_ρ with branching only at the intersection with the proper transforms of the 30 planes above, as follows:

$$- P_{ij}: E_\rho \times E_\rho \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \text{ a Galois } \mathbb{Z}/3\mathbb{Z}\text{-quotient;}$$

– $Q_{ij}: E_\rho \times E_\rho \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the product of two double covers branched at $0, 1, \rho, \rho^2$.

This supports by Lemma 2.5.1 the idea that this could be the compactification locus of $X_{(1,1,1,3)}(\sqrt{-3})$.

5 The Coble variety \mathcal{Y}

5.1 Arithmetic quotients of domains of type \mathbf{IV}_n

Let V be a k vector space of dimension n , k a totally real field, and b a bilinear symmetric form on V . Let $G(V, b)$ be the symmetry group, and $G_{\mathbb{Q}} = \text{Res}_{k|\mathbb{Q}}G(V, b)$ the \mathbb{Q} -group it defines. We assume that $G_{\mathbb{Q}}$ is of hermitian type, so that for every infinite prime ν of k the signature of b_ν is $(n-2, 2)$ or b_ν is definite. $G_{\mathbb{Q}}$ is (absolutely) simple (defines an irreducible domain) only if $k = \mathbb{Q}$ (or if b_ν is definite for all but a single ν , in which case the \mathbb{Q} -group is anisotropic, but we will not consider this situation), and the corresponding real group gives rise to a bounded symmetric domain only if b has Witt index 2. This is the case we consider here. The classification of such forms is well-known; since we require the Witt index to be 2, two such forms b and b' are equivalent over $\mathbb{Q} \iff \det(b) = \det(b')$, where $\det(b)$ is to be viewed as an element of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$.

Now let $\mathcal{L} \subset V$ be a (maximal) lattice, and let $G_{\mathcal{L}}$ be the arithmetic group it defines, $\Gamma \subset G_{\mathcal{L}}$ a subgroup of finite index. We first remark on the moduli interpretation of the arithmetic quotient X_Γ .

Proposition 5.1 *X_Γ is a moduli space of (pure) Hodge structures of weight 2 on V with $h^{2,0} = 1$ (and $h^{1,1} = \dim(X_\Gamma)$) with respect to the lattice $\mathcal{L} \subset V$.*

Proof: The symmetry group of such a Hodge structure is of real type $SO(n-2, 2)$, and $G(V, b)$ is a \mathbb{Q} -form in which $G_{\mathcal{L}}$ is an arithmetic subgroup. Since the corresponding “period” (i.e., position of the varying complex subspace $H^{1,1}$ in $H_{\mathbb{C}}^2$) is clearly the same exactly when the two periods differ by an element of $G_{\mathcal{L}}$, while Γ defines a level structure of some kind, the result follows. \square

This proposition is often used in the study of polarised K3-surfaces, which have a pure Hodge structure of type $(1, 19, 1)$. In fact, for each polarisation degree (i.e., the number C^2 for the ample divisor C on the K3-surface which gives the projective embedding) $2e$, ($e \geq 1$) one has an arithmetic group Γ_e such that the arithmetic quotient X_{Γ_e} is the moduli space of K3 surfaces with the given polarisation. Recall the *Picard number* ρ is the rank of the group of algebraic cycles, i.e., $\rho = \text{rk}_{\mathbb{Z}}H^2(S, \mathbb{Z}) \cap H^{1,1}$. Then one has the following.

Proposition 5.2 *Let S be a K3 surface with $\rho =$ the Picard number of S . Then the dimension of the moduli space of K3's which are in the family preserving the lattice of algebraic cycles $H^2(S, \mathbb{Z}) \cap H^{1,1}$ is $20 - \rho$.*

Proof: Recall that for a K3 surface $H^1(S, \Theta) \cong H^1(S, \Omega^1)$, so $H^{1,1}(S)$ may be viewed as the tangent space of the local deformation space, which should be thought of as a varying complex subspace of $H^2(S, \mathbb{C})_{\text{prim}}$, while $H^2(S, \mathbb{Z})$ is fixed. Let $\mathcal{A} = H^2(S, \mathbb{Z}) \cap H^{1,1}$ be the lattice of algebraic cycles, $\mathcal{T} = H^2(S, \mathbb{Z}) \cap (H^{2,0}(S) \oplus H^{0,2}(S))$ the lattice of transcendental cycles. We have $\text{rk}_{\mathbb{Z}}\mathcal{A} = \text{rk}_{\mathbb{Z}}(H^2(S, \mathbb{Z})) - \text{rk}_{\mathbb{Z}}\mathcal{T}$ in general and $\text{rk}_{\mathbb{Z}}\mathcal{A} = \rho$ by assumption, so $\rho = 22 - \text{rk}_{\mathbb{Z}}\mathcal{T}$, while the moduli space is defined by the group $G(V', b')$, where $V' = \mathcal{A}^\perp \otimes \mathbb{Q}$, since we are requiring \mathcal{A} to be preserved. (Recall that for an algebraic cycle \mathcal{C} the integral $\int_{\mathcal{C}} \omega$ over the holomorphic two-form ω vanishes, hence the algebraic cycles contribute nothing to the periods). Thus $G(\mathbb{R}) = SO(20 - \rho, 2)$, giving rise to a domain of type $\mathbf{IV}_{20-\rho}$. \square

Of course in this particular case, the lattice $\mathcal{L} \subset V$ is very special; the “intersection form” b restricted to \mathcal{L} is even and unimodular, and as is well-known, decomposes as

$$\mathcal{L} \cong \langle -2e \rangle \oplus \mathbf{H}^2 \oplus \mathbf{E}_8^2, \quad (88)$$

where \mathbf{H} is the two-dimensional hyperbolic lattice, and \mathbf{E}_8 is the root lattice of type \mathbf{E}_8 . Let us remark that the compactification of these arithmetic quotients has been carried out in the thesis [S], but we will not need this. We will very quickly describe a particularly interesting family of K3 surfaces which has been thoroughly studied by Yoshida and his collaborators, see [MSY] for details on all matters here.

5.2 A four-dimensional family of K3's

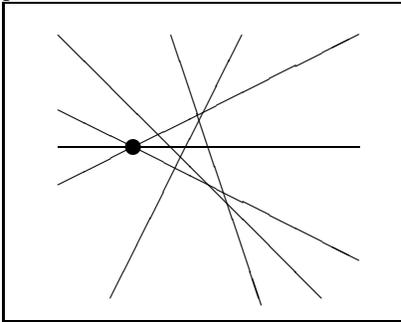
The family of K3 surfaces to be described here is the set of surfaces which are double covers of \mathbb{P}^2 branched along the union of six disjoint lines. Recall that there is a 19-dimensional family of K3 surfaces which are double covers of the plane branched along a sextic curve; they are smooth as long as the sextic is smooth, and generically have Picard number $\rho = 1$. An arrangement of six lines in \mathbb{P}^2 is a maximally singular sextic; there are 15 intersection points of the six lines (if they are in general position), and each such gives rise to an A_1 -singularity on the double cover. Resolving the 15 double points introduces 15 exceptional curves with self intersection number -2 , so together with the pullback of the generic line, this gives 16 independent cycles on the surface: $\rho = 16$. Hence the transcendental lattice \mathcal{T} has rank 4, so by Proposition 5.2, the moduli space is four-dimensional. Let

$$\Gamma = \{g \in G(\mathcal{T}_{\mathbb{R}}, Q) (\cong SO(4, 2)) \mid g(\mathcal{T}) \subset \mathcal{T}\}, \tag{89}$$

where Q is the intersection form on $H^2(S, \mathbb{Z})$, extended to \mathbb{R} , then restricted to $\mathcal{T}_{\mathbb{R}}$. This is clearly an arithmetic subgroup, and by Proposition 5.1, the arithmetic quotient $X_{\Gamma} = \Gamma \backslash \mathcal{D}$ is the four-dimensional moduli space. We list some of the interesting loci for this family. Let L be the given arrangement, $L = l_1 \cup \dots \cup l_6$, and let $t_p :=$ the number of p -fold points of the arrangement, i.e., the number of points at which p of the lines meet (see (8)), and let $\pi : S \rightarrow \mathbb{P}^2$ denote the (singular) double cover.

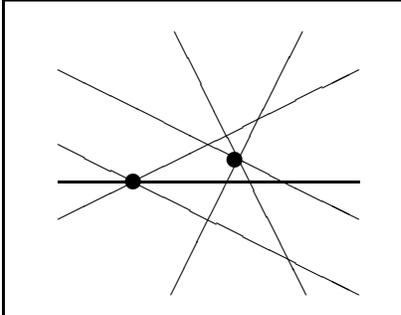
5.2.1 Three-dimensional loci

- 1) Suppose there is a conic which is *tangent* to all six lines. Then the inverse image of this quadric is a \mathbb{P}^1 , which, as is easily checked, has self-intersection number $4 - 6 = -2$, so the double cover has 16 exceptional cycles, hence $\rho = 17$. It is in fact easy to see that the surface S is in this case a classical Kummer surface, i.e., a quartic surface in \mathbb{P}^3 with 16 nodes which is the Kummer variety of a principally polarised abelian surface A_S . The projection from a node gives the double cover $\pi : S \rightarrow \mathbb{P}^2$, and the tangent conic is the image of the (blown up) node used to project. The abelian surface is the Jacobian of a genus 2 curve, and this curve is the double cover of the conic, *branched at the six points of tangency*. This is well-known.
- 2) If $t_3 = 1, t_2 = 12$, then the threefold point induces an A_2 -singularity on the double cover which is resolved by two \mathbb{P}^1 's, so there are now $2+12$ exceptional \mathbb{P}^1 's and the hyperplane section. We have the following picture.



There are in fact three more exceptional \mathbb{P}^1 's, which are the inverse image on the double cover of the three lines which pass through the triple point and one of the three double points not lying on a line through the triple point. It is easy to see that these three double points are independent parameters of such arrangements, so this defines a three-dimensional family, so by Proposition 5.2, we have $\rho = 17$ for the generic member of this family.

5.2.2 Two-dimensional loci

- 3)  If $t_3 = 2, t_2 = 9$, there are two possibilities. Suppose first that the two threefold points do *not* lie on one of the six lines. Then we have the picture to the left. This gives rise to two isolated A_2 -singularities. The inverse image of the line joining the two threefold points is also an exceptional \mathbb{P}^1 . In this case the generic double cover has $\rho = 18$, and as parameters one can take two double ratios: consider two of the lines l_1, l_2 , both passing through one of the threefold points p ; the three intersection points with the other lines, together with p , give four points on each line – hence two double ratios.

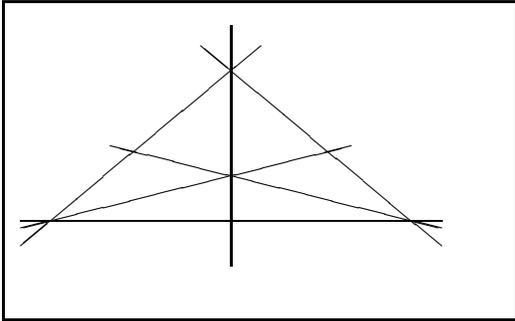
- 4) It may also occur that both threefold points lie on a line, but in this case we also have $\varrho = 18$, i.e., a two-dimensional family.

5.2.3 One-dimensional loci

- 5) If $t_4 = 1$, then the double cover has an *elliptic* singularity over the point, so is not K3. Hence this is a genuine *degeneration* of the K3, i.e., belongs to the boundary of the compactification. It turns out that then a line must also be double, so that the double cover has two components.
- 6) As a further specialisation of 4) it may happen that there are three triple points. Since four of the lines may be chosen fixed (for example $x_0 = x_1 = x_2 = 0$, $x_0 - x_1 = 0$), there is only one modulus, given for example by the intersection point of the two variable lines. Here we have $\varrho = 19$.

5.2.4 Zero-dimensional loci

- 7) If three lines are taken, each *double*, then the double cover splits into two copies of \mathbb{P}^2 . This is in the closure of the set of degenerations of type 5).
- 8) The arrangement is the complete quadrilateral. The picture is:



It is known that the *Fermat* cover (not the double cover) of this arrangement is Shioda's elliptic modular surface of level 4, $S(4)$, so it follows that the double cover is isogenous to $S(4)$, i.e., a quotient of $S(4)$ by a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. This is the most special K3 surface in the family and has $\varrho = 20$.

5.2.5 Level 2 structure

Now consider, in addition to the above data, a level 2 structure. Geometrically this amounts to fixing an *order* of the six lines. In terms of the lattice \mathcal{T} it is not so easy to see what it means. In [MSY] it is shown by explicit computation that the subgroup $\Gamma(2)$ is the group generated by reflections on the “roots” of \mathcal{T} , that is the integral elements of norm -2 . Furthermore it is shown there that Γ is generated by the reflections on the elements of norm -2 or -4 , and that $\Gamma/\Gamma(2) \cong \Sigma_6 \times \mathbb{Z}/(2)$. Hence by the results 2.7.1, 2.7.7, 2.8.2 of [MSY] we have

Proposition 5.3 *The arithmetic quotient $\Gamma(2)\backslash\mathcal{D}$ is the moduli space for K3 surfaces which are double covers of \mathbb{P}^2 , branched over an ordered set of six lines.*

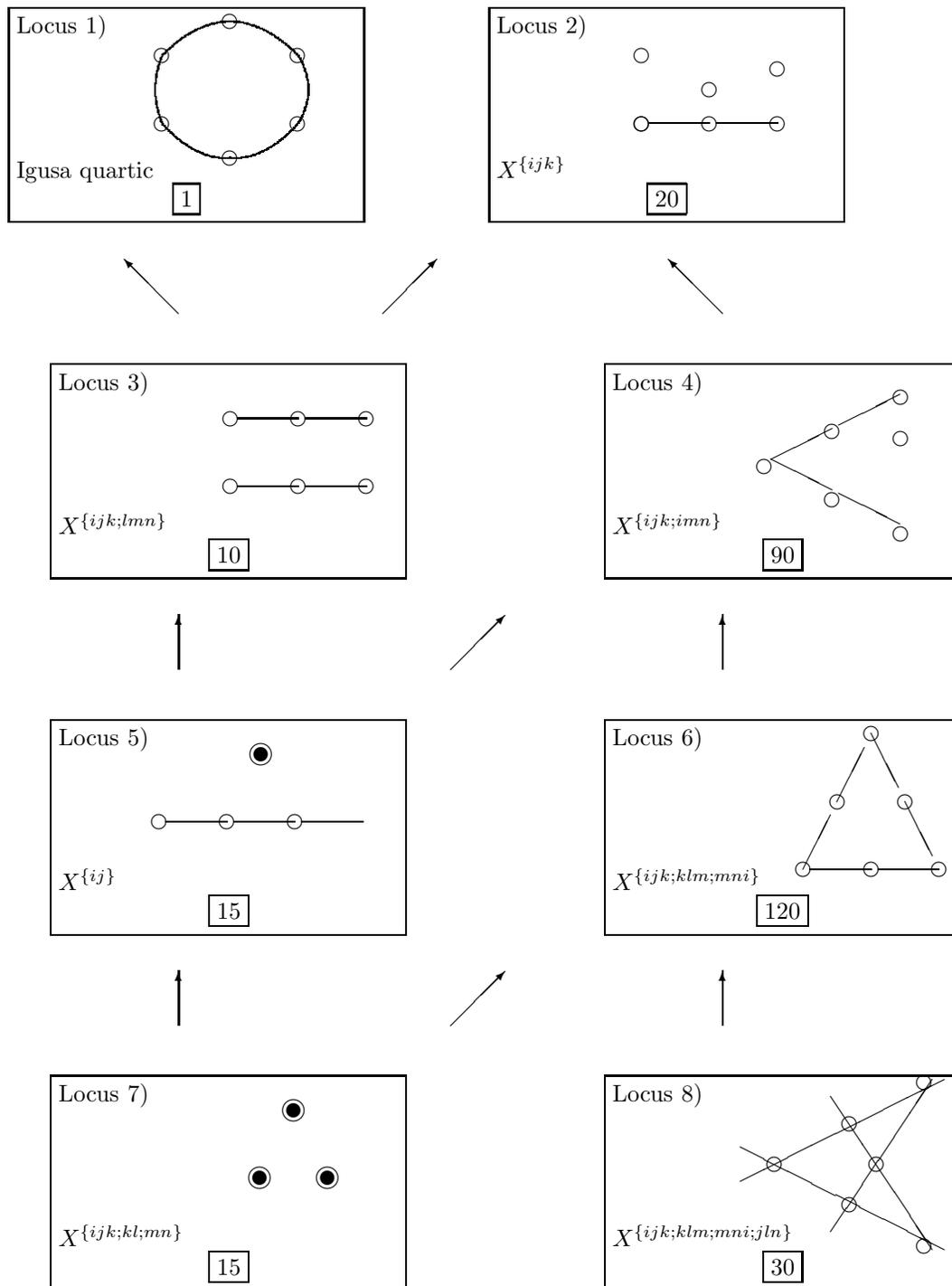
We refer the reader to [MSY] for a detailed description of the loci described above, of the periods and of the corresponding Picard-Fuchs equations (and much more). We give in Table 4 a description of the loci, giving the dual graph of the six lines (i.e., a vertex for each line, two vertices lying on a line \iff the corresponding lines meet), as well as the number of loci, and the names given to them in [MSY].

We now give an explicit projective description of the Baily-Borel compactification of the arithmetic quotient $\Gamma(2)\backslash\mathcal{D}$ of Proposition 5.3. All the facts presented here were proved originally by Coble [C] or by Yoshida and his collaborators in [MSY]. We have the four-dimensional family of K3 surfaces just discussed, defined in terms of a set of (ordered) six lines in the plane. Dual to the six lines are six points, and so the relation with the moduli space of cubic surfaces is evident. Let two ordered sets of six lines, (l_1, \dots, l_6) , (l'_1, \dots, l'_6) be given.

Definition 5.4 The two sets of lines (l_1, \dots, l_6) , (l'_1, \dots, l'_6) are said to be *associated*, if the following relation holds. Since the set (l_1, \dots, l_6) is ordered, we can form two triangles,

$$\Delta(l_1, l_2, l_3), \quad \Delta(l_4, l_5, l_6);$$

Table 4: Loci of a four-dimensional family of K3 surfaces



The notations $X^{\{ijk;klm;mni;jln\}}$, etc, are taken from [MSY]; the arrows indicate inclusions among the various loci. The symbol \bullet means a double line. The number of each kind of loci is indicated by \boxed{x} ; the dimensions are three in the top row down to zero in the last row. Locus 1) is where the six points lie on a conic, while the 20 $X^{\{ijk\}}$ are the loci where there are three of the six points on a line. The 15 $X^{\{ijk;kl;mn\}}$ lie on the boundary of the moduli space, while the 30 $X^{\{ijk;klm;mni;jln\}}$ lie in “the farthest interior” of the domain. A more complete description is given in Corollary 5.10.

these two triangles have together six vertices, which come equipped with a numbering, say (p_1, \dots, p_6) , and these correspond dually to another ordered set of six lines, $(l_{p_1}, \dots, l_{p_6})$. Then (l_1, \dots, l_6) and (l'_1, \dots, l'_6) are associated, if: $(l_{p_1}, \dots, l_{p_6}) = (l'_1, \dots, l'_6)$, as a set of six ordered lines.

Of course, starting with two sets of ordered six points, one can define in the same way the notion of association. Since, as abstract moduli spaces, the space of ordered sets of six lines is the “same” (by duality) as the set of ordered sets of six points, we see that we are dealing here with the space of sets of six ordered points in \mathbb{P}^2 . This problem was dealt with in the papers of Coble [C], and has been given a modern treatment in [DO]. It can be described as follows. The relevant moduli space is easy to describe: let (p_1, \dots, p_n) be a set of n points in \mathbb{P}^k ; this is represented by M , the $n \times (k+1)$ matrix whose i^{th} column gives the coordinates of the point p_i . The moduli space is then the GIT quotient

$$\mathbf{P}_n^k = GL(k+1) \backslash M(n, k+1) / (\mathbb{C}^*)^n. \quad (90)$$

By taking the set of semistable points in $M(n, k+1)$ the above quotient is compact, although singular. It is classical that \mathbf{P}_6^1 is a threefold whose compactification can be identified with a cubic threefold in \mathbb{P}^4 with ten ordinary double points, which is just the Segre cubic \mathcal{S}_3 . Note that the similar moduli problem, namely six points on a conic in \mathbb{P}^2 , is realised by the Igusa quartic \mathcal{I}_4 , so these are very closely related, but not identical moduli problems.

Our interest here is in \mathbf{P}_6^2 , a fourfold. In this case we may represent elements by matrices

$$\mathbf{P}_n^k \ni M = \begin{bmatrix} 1 & 0 & 0 & 1 & x & w \\ 0 & 1 & 0 & 1 & y & z \\ 0 & 0 & 1 & 1 & u & u \end{bmatrix}, \quad (91)$$

and as Coble shows, the map $\mathbf{P}_6^2 \rightarrow \mathbb{P}^4$, $M \mapsto [x : y : w : z : u]$ is a birational map (it is clear that \mathbf{P}_6^2 is rational, this map simply gives an explicit birationalisation). The GIT theory here consists of finding G -invariant functions on \mathbf{P}_6^2 , and these turn out to be generated by 3×3 minors of M .

In terms of the matrix M the process of association can be described as follows. Each such matrix M determines a second one: since the six points are ordered, one can define six lines by $l_{12} = \overline{p_1 p_2}$, $l_{13} = \overline{p_1 p_3}$, $l_{23} = \overline{p_2 p_3}$, $l_{45} = \overline{p_4 p_5}$, $l_{46} = \overline{p_4 p_6}$, $l_{56} = \overline{p_5 p_6}$; these six lines determine dually six points, whose coordinates are then brought into the normal form given above. It turns out that the entries of the second matrix are determined by the fact that the maximal minors are proportional to the maximal minors of the first. More precisely, if we let (ijk) denote the 3×3 minor of M which is given by the columns i, j, k , and if we let M' be the associated matrix, $(ijk)'$ the corresponding minor, then the minors of M and M' are related by:

$$(123)(145)(246)(356) = (124)'(135)'(236)'(456)'. \quad (92)$$

Now association is an involution on \mathbf{P}_6^2 , and one can take the *quotient* by this involution.

Definition 5.5 Let \mathcal{Y} be the double cover of \mathbb{P}^4 branched along the Igusa quartic \mathcal{I}_4 , $\pi : \mathcal{Y} \rightarrow \mathbb{P}^4$.

Clearly \mathcal{Y} will be singular precisely along the singular locus of \mathcal{I}_4 , i.e.,

Lemma 5.6 *The singular locus of \mathcal{Y} consists of 15 lines, the inverse images of the 15 singular lines of \mathcal{I}_4 .*

Theorem 5.7 ([DO], Example 4, p. 37) *The moduli space of six ordered points in \mathbb{P}^2 is equal to the double cover \mathcal{Y} , and the double cover involution on \mathcal{Y} coincides with the association involution on \mathbf{P}_6^2 .*

In other words, a set (p_1, \dots, p_6) is *associated to itself*, if and only if the six points lie on a conic in \mathbb{P}^2 .

Consider one of the hyperplanes H in \mathbb{P}^4 , $H = \{\theta_m^4 = 0\}$ of Proposition 3.9. Since H is *tangent* to \mathcal{I}_4 , the inverse image $\pi^{-1}(H)$ in \mathcal{Y} will *split into two copies* of \mathbb{P}^3 . In this way, we get a union of 20 \mathbb{P}^3 's on \mathcal{Y} ,

Lemma 5.8 *The inverse images $\pi^{-1}(H)$ of the tangent hyperplanes $H = \{\theta_m^4 = 0\}$ consist of two copies each of \mathbb{P}^3 , and these two \mathbb{P}^3 's on \mathcal{Y} meet in the quadric surface which is the inverse image under π of the quadric on \mathcal{I}_4 to which H is tangent. This gives a total of 20 such \mathbb{P}^3 's on \mathcal{Y} .*

A resolution of singularities of \mathbf{P}_6^2 is affected by resolving the Igusa quartic by blowing up the ideal of the 15 lines; this is the map ϱ_2 of Theorem 3.11. Let $\widehat{\mathbf{P}}_6^2$ denote this desingularisation $\widehat{\mathbf{P}}_6^2 \rightarrow \mathbf{P}_6^2$. On $\widehat{\mathbf{P}}_6^2$ we have a set of 36 divisors, the *discriminant locus*, the proper transforms of the Igusa quartic, the 20 \mathbb{P}^3 's and the 15 exceptional divisors of the blow up.

It is clear how this variety is the moduli space of cubic surfaces: blow up \mathbb{P}^2 in the six points, and embed by the linear system of cubic curves through the six points. The ordering of the six points of course determines a marking of the 27 lines in the well-known manner. The symmetry group of $\widehat{\mathbf{P}}_6^2$ is $\Sigma_6 \times \mathbb{Z}/2\mathbb{Z}$; although the Weyl group $W(E_6)$ acts birationally on it, the action is not regular. For that it is necessary to modify $\widehat{\mathbf{P}}_6^2$ even more. Dolgachev mentions in [DO] that he suspects it is sufficient to blow up $\widehat{\mathbf{P}}_6^2$ in the intersection of the 36 divisors.

One of the many things proved in [MSY] is the following.

Theorem 5.9 *The variety \mathcal{Y} is the Baily-Borel compactification of the arithmetic quotient $\Gamma(2)\backslash\mathcal{D}$ of Proposition 5.1.*

The proof given in [MSY] of this fact simply (!) calculates the image of the period map, and in determining when the periods lie on the boundary of the period domain \mathcal{D} , the authors find that this locus coincides with the set of K3 surfaces whose set of six lines correspond to those singularities of Lemma 5.6 of \mathcal{Y} .

Corollary 5.10 *The Loci 5) and 7) of Table 4 are the inverse images on \mathcal{Y} of the 15 singular lines and 15 singular points, respectively, of the branch locus \mathcal{I}_4 . The Loci 3) of Table 18 are the inverse images of the ten special hyperplane sections of Lemma 5.8, i.e., the quadrics. The loci 2) of Table 18 are the 20 \mathbb{P}^3 's of Lemma 5.8, and Locus 1) is just the branch locus of the double cover.*

Part II

A Gem of the modular universe

6 The Weyl group $W(E_6)$

6.1 Notations

We use the same notation as above for the 27 lines on a cubic surface in \mathbb{P}^3 : $a_1, \dots, a_6, b_1, \dots, b_6, c_{12}, \dots, c_{56}$. The 36 double sixes are:

$$N = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}, \quad (1)$$

$$N_{ij} = \begin{bmatrix} a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} \\ a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} \end{bmatrix}, \quad (15)$$

$$N_{lmn}^3 = \begin{bmatrix} a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} \\ c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n \end{bmatrix} \quad (20).$$

The 45 tritangents are:

$$\begin{aligned} (ij) &= \langle a_i b_j c_{ij} \rangle, \quad i \neq j \quad (30) \\ (ij.kl.mn) &= \langle c_{ij} c_{kl} c_{mn} \rangle \quad (15). \end{aligned} \quad (93)$$

Two double sixes are *syzygetic* if they contain four lines in common, for example:

$$N \text{ and } N_{12} \text{ have } a_1, a_2, b_1, b_2 \text{ in common,}$$

and *azygetic* if they have six lines in common, for example:

$$N \text{ and } N_{456} \text{ have } a_1, a_2, a_3, b_4, b_5, b_6 \text{ in common.}$$

³here we switch notations from N_{ijk} in equation (1) to N_{lmn} for convenience

Two azygetic double sixes have six lines in common and contain 12 other lines; these 12 lines form another double six, azygetic with respect to both, for example N, N_{123}, N_{456} . Such triples are referred to as triples of azygetic double sixes or, because of the interpretation in terms of tritangents, a trihedral pair. Each double six is syzygetic to 15 others, forming 270 such pairs, and azygetic to 20 others, forming 120 triples. Our notation for the 120 triples are:

$$\begin{aligned} \{ijk\} &= \langle N, N_{ijk}, N_{lmn} \rangle, & (10) \\ \{i\dot{j}.jk\} &= \langle N_{ij}, N_{ik}, N_{jk} \rangle, & (20) \\ \{i\dot{j}.kl\} &= \langle N_{ij}, N_{ikl}, N_{jkl} \rangle & (90). \end{aligned}$$

We recognize these as the trihedral pairs of (2) under the correspondence

$$\begin{aligned} \begin{bmatrix} a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j \\ c_{ik} & a_k & b_i \end{bmatrix} &\longleftrightarrow \langle N_{ij}, N_{ik}, N_{jk} \rangle, & \begin{bmatrix} c_{il} & c_{jm} & c_{kn} \\ c_{mn} & c_{ik} & c_{jl} \\ c_{jk} & c_{ln} & c_{im} \end{bmatrix} &\longleftrightarrow \langle N, N_{ijk}, N_{lmn} \rangle, \\ & & \begin{bmatrix} a_i & b_j & c_{ij} \\ b_l & a_k & c_{kl} \\ c_{il} & c_{jk} & c_{mn} \end{bmatrix} &\longleftrightarrow \langle N_{ij}, N_{ikl}, N_{jkl} \rangle. \end{aligned}$$

Hence the triads of trihedral pairs discussed there are expressed in condensed form as follows:

$$[ijk.lmn] = \begin{bmatrix} N_{ij} & N_{jk} & N_{ik} \\ N_{lm} & N_{mn} & N_{ln} \\ N & N_{ijk} & N_{lmn} \end{bmatrix}, \quad (10)$$

$$[i\dot{j}.kl.mn] = \begin{bmatrix} N_{ij} & N_{ikl} & N_{jkl} \\ N_{kl} & N_{kmn} & N_{lmn} \\ N_{mn} & N_{nij} & N_{mij} \end{bmatrix}, \quad (30).$$

The group of incidence preserving permutations of the 27 lines, a group of order 51840, can be generated by the following six operations:

$$(i, i+1), \quad i = 1, \dots, 5: \text{ transposition of the indices,}$$

and

$$(123) : \text{map } N \mapsto N_{123},$$

and the graph of this presentation is shown in Figure 4. This is the graph whose vertices correspond to

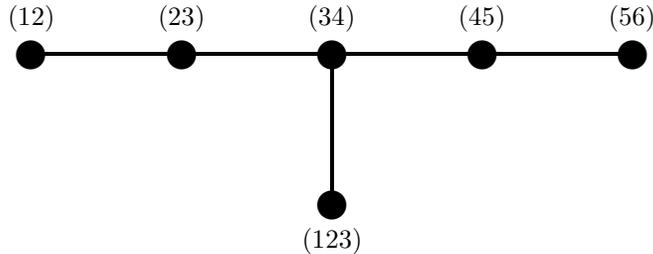


Figure 4: The graph of the group of the permutations of the 27 lines

generators, two vertices A, B being connected if $ABA=BAB$ and not connected if $AB=BA$.

6.2 Roots

Let \mathfrak{t} be a maximal abelian subalgebra of the compact Lie algebra $\mathfrak{e}_{6,u}$ over \mathbb{R} , i.e., $\mathfrak{t} \cong \mathbb{R}^6$. Let x_1, \dots, x_6 be coordinates such that the root forms of E_6 are:

$$(40) \quad \pm(x_i \pm x_j), \quad 1 \leq i < j \leq 5$$

$$(32) \quad \pm \frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 + x_6), \quad \text{even number of “-” signs inside the parenthesis.}$$

(Note that in Bourbaki notation, our variables $x_i = \varepsilon_i$, $i = 1, \dots, 5$, while our coordinate x_6 is denoted $\varepsilon_8 - \varepsilon_7 - \varepsilon_6$ there). The 36 positive root forms are given by $\pm x_i + x_j$ and $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 + x_6)$, and they correspond to the 36 double sixes of the 27 lines on a cubic surface. We use the following notations for these forms

$$\begin{aligned}
h &= \frac{1}{2}(x_1 + \dots + x_6), \\
h_{1j} &= x_{j-1} - \frac{1}{2}(x_1 + \dots + x_5 - x_6), \quad j = 2, \dots, 6 \\
h_{jk} &= -x_{j-1} + x_{k-1}, \quad 1 \neq j < k \\
h_{1jk} &= x_{j-1} + x_{k-1}, \quad j, k = 2, \dots, 6 \\
h_{jkl} &= +x_{j-1} + x_{k-1} + x_{l-1} - \frac{1}{2}(x_1 + \dots + x_5 - x_6), \quad j, k, l \neq 1.
\end{aligned} \tag{96}$$

The Weyl group of E_6 is generated by the reflections on these 36 hyperplanes; we denote these reflections by s , s_{ij} , and s_{ijk} . As a system of simple roots we take :

$$\begin{aligned}
\alpha_1 &= -\frac{1}{2}(-x_1 + \dots + x_5 - x_6) = h_{12} \\
\alpha_2 &= x_1 + x_2 = h_{123} \\
\alpha_3 &= -x_1 + x_2 = h_{23} \\
\alpha_4 &= -x_2 + x_3 = h_{34} \\
\alpha_5 &= -x_3 + x_4 = h_{45} \\
\alpha_6 &= -x_4 + x_5 = h_{56}.
\end{aligned} \tag{97}$$

Then the Dynkin diagram is as shown in Figure 5; we recover Figure 4 by replacing α_i by the corresponding reflection s , s_{ij} , s_{ijk} on the hyperplanes where h , h_{ij} , h_{ijk} , respectively, vanish. This shows clearly the isomorphism of $W(E_6)$ and the group of the permutations of the 27 lines,

$$\text{Aut}(\mathcal{L}) \cong W(E_6).$$

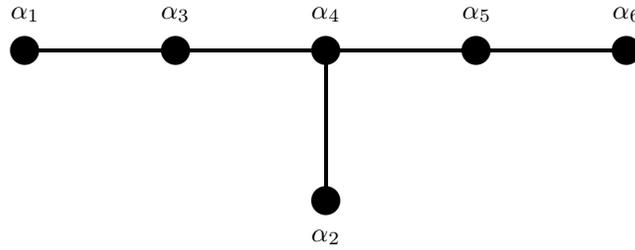


Figure 5: The Dynkin diagram of the Weyl group of E_6

The action of the reflections on the root forms can be described as follows:

$s(h_{ij}) = h_{ij}$	$s(h_{ijk}) = h_{lmn}$	$s_{ijk}(h) = h_{lmn}$	$s_{ij}(h) = h$
$s_{ijk}(h_{lmn}) = h$	$s_{ijk}(h_{kmn}) = h_{kmn}$	$s_{ijk}(h_{jkn}) = h_{in}$	$s_{ijk}(h_{ln}) = h_{ln}$
$s_{ijk}(h_{kn}) = h_{ijn}$	$s_{ijk}(h_{jk}) = h_{jk}$	$s_{ij}(h_{klm}) = h_{klm}$	$s_{ij}(h_{jlm}) = h_{ilm}$
$s_{ij}(h_{ijm}) = h_{ijm}$	$s_{ij}(h_{ij}) = h_{ij}$	$s_{ij}(h_{jk}) = h_{ik}$	$s_{ij}(h_{lk}) = h_{lk}$

6.3 Vectors

The Killing form of E_6 , a quadratic invariant, can be calculated as the sum of the squares of all roots, and evaluates to (a constant times):

$$I_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + \frac{1}{3}x_6^2.$$

Table 5: The arrangement in \mathbb{P}^5

$N(\mathcal{O})$	A_1	A_1^2	A_2	A_1^3	$A_{1,2}$	A_3	$A_{1,2,2}$	A_2^2	$A_{1,3}$	A_4	D_4	$A_{1,2^2}$	$A_{1,4}$	A_5	D_5
\mathcal{O}	k	$t_2(3)$	$t_3(3)$	$t_3(2)$	$t_4(2)$	$t_6(2)$	$t_5(1)$	$t_6(1)$	$t_7(1)$	$t_{10}(1)$	$t_{12}(1)$	t_7	t_{11}	t_{15}	t_{20}
#	36	270	120	540	720	270	1080	120	540	216	45	360	216	36	27
$t(4)$	1	15	10	45	80	45	150	20	105	60	15	70	66	15	15
$t_2(3)$		1	0	6	8	3	28	4	18	12	3	20	20	6	7
$t_3(3)$			1	0	6	9	9	2	18	18	6	6	18	6	9
$t_3(2)$				1	0	0	6	0	3	0	1	6	6	1	3
$t_4(2)$					1	0	3	1	3	3	0	4	6	3	3
$t_6(2)$						1	0	0	2	4	2	0	4	2	5
$t_5(1)$							1	0	0	0	0	2	2	0	1
$t_6(1)$								1	0	0	0	3	0	3	0
$t_7(1)$									1	0	0	0	2	1	1
$t_{10}(1)$										1	0	0	1	1	2
$t_{12}(1)$											1	0	0	0	3

With respect to the Killing form we have the vectors dual to the root forms:

$$\begin{aligned}
H &= \frac{1}{2}(1, 1, 1, 1, 1, 3); \\
H_{1j} &= -\frac{1}{2}(1, \dots, -1, \dots, -3), & 1 \text{ in the } (j-1)^{st} \text{ spot, } j = 2, \dots, 6; \\
H_{jk} &= -\frac{1}{2}(0, \dots, 1, \dots, -1, \dots, 0), & \pm 1 \text{ in the } (j-1)^{st}, (k-1)^{st} \text{ spot, } 1 < j < k \leq 6; \\
H_{1jk} &= \frac{1}{2}(0, \dots, 1, \dots, 1, \dots, 0), & 1 \text{ in the } (j-1)^{st}, (k-1)^{st} \text{ spot, } 1 < j < k \leq 6; \\
H_{jkl} &= -\frac{1}{2}(1, -1, \dots, -1, \dots, -1, \dots, -3), & 1's \text{ in the } (j-1), (k-1), (l-1) \text{ spots,} \\
& & 1 < j < k < l \leq 6;
\end{aligned} \tag{98}$$

which may be thought of as the root vectors (of the positive roots; the negative roots have a “-” sign in front).

As is well-known, there is also a set of 27 fundamental weights which form an orbit of $W(E_6)$, namely:

$$\begin{aligned}
a_1 &= -\frac{2}{3}x_6; & a_j &= x_{j-1} - \frac{1}{2}(x_1 + \dots + x_5 + \frac{1}{3}x_6); \\
b_1 &= \frac{1}{2}(x_1 + \dots + x_5 - \frac{1}{3}x_6); & b_j &= x_{j-1} + \frac{1}{3}x_6; \\
c_{1j} &= -x_{j-1} + \frac{1}{3}x_6; & c_{ij} &= -x_{j-1} - x_{i-1} + \frac{1}{2}(x_1 + \dots + x_5 - \frac{1}{3}x_6).
\end{aligned} \tag{99}$$

These form the $W(E_6)$ orbit of the fundamental weights denoted $\bar{\omega}_1$ and $\bar{\omega}_6$ in Bourbaki, which are just our $-a_1$ and b_6 , respectively. Note that the following relation holds:

$$\sum_{i=1}^6 a_i = -3h = -3\left(\sum_{i=1}^6 x_i\right) = -\sum_{i=1}^6 b_i. \tag{100}$$

Also note that the a_i and b_i are related by

$$b_i = a_i - \frac{1}{3}(a_1 + \dots + a_6). \tag{101}$$

The corresponding vectors which are dual with respect to the Killing form are:

$$\begin{aligned}
A_1 &= (0, \dots, 0, -2); & A_j &= \frac{1}{2}(-1, \dots, +1, \dots, -1) & +1 \text{ in the } j-1 \text{ spot;} \\
B_1 &= \frac{1}{2}(1, \dots, 1, -1); & B_j &= (0, \dots, 1, \dots, 1) & +1 \text{ in the } j-1 \text{ spot;} \\
C_{1j} &= (0, \dots, -1, \dots, 1); & C_{ij} &= \frac{1}{2}(1, \dots, -1, \dots, -1, \dots, -1) & -1 \text{ in the } j-1, i-1 \text{ spots.}
\end{aligned} \tag{102}$$

6.4 The arrangement defined by $W(E_6)$

The 36 hyperplanes in \mathbb{P}^5 defined by the vanishing of the 36 root forms form the arrangement $\mathcal{A}(\mathbf{E}_6)$ of (7). For later reference we give the combinatorial data of the arrangement here. We denote as in (8) a \mathbb{P}^m through which k of the hyperplanes pass by $t_k(m)$. For the normalisers we use the notation A_{i^k, j^l} for $A_i^k \times A_j^l$. The data of the arrangement is given in Table 5.

6.5 Special Loci

In Table 6 we give a list of special loci which will be particularly important in what follows, so we give a brief description of each.

Table 6: Special loci in \mathbb{P}^5

#	space	Symmetry	$N(\mathcal{O})$	notation in Table 5
36	\mathbb{P}^4	A_5	A_1	—
120	\mathbb{P}^3	D_4	A_2	$t_3(3)$
120	\mathbb{P}^1	A_2	$A_2 \times A_2$	$t_6(1)$
216	\mathbb{P}^1	A_2	A_4	$t_{10}(1)$
45	\mathbb{P}^1	A_1	D_4	$t_{12}(1)$
36	point	—	A_5	t_{15}
27	point	—	D_5	t_{20}

6.5.1 36 \mathbb{P}^4 's

In each of the 36 hyperplanes given by the vanishing of one of the 36 forms (96), h say, the induced group is Σ_6 , and as a reflection group on \mathbb{P}^4 it defines a projective arrangement of 15 planes; since each double six is syzygetic to 15 and azygetic to 20 others, there are 15 hyperplanes through which one of the other 35 intersect h , and ten planes through which two others of the 35 meet h . We immediately recognize this geometry as that in \mathbb{P}^4 discussed in the first part of the paper. The 15 hyperplanes are the 15 \mathcal{H}_{ij} of (38), each of which cuts out three planes on \mathcal{S}_3 , and the ten are the hyperplanes mentioned in (50) and Proposition 3.9. These in turn are the dual hyperplanes to the ten nodes on \mathcal{S}_3 .

6.5.2 120 \mathbb{P}^3 's

These \mathbb{P}^3 's correspond to the 120 triples of azygetic double sixes, i.e., each is cut out by three of the 36 hyperplanes of 6.5.1. In each such hyperplane, these \mathbb{P}^3 's correspond to the ten hyperplanes in h just mentioned, given by the K_{ijk} of (50). Each of these contains 15 planes, and one can check that these are just the faces and symmetry planes of a cube. The six lines in K_{ijk} which are the singular locus $\mathcal{I}_4 \cap K_{ijk}$, are easily identified with the six 12-fold lines $t_{12}(1)$ which are contained in K_{ijk}^4 , and the nine points t_{20} contained in K_{ijk} are the intersection points of those six lines⁵. Equations of the 120 \mathbb{P}^3 's are given by a triple of azygetic double sixes, e.g., by $\langle h, h_{ijk}, h_{lmn} \rangle$.

6.5.3 120 \mathbb{P}^1 's

The 120 lines correspond exactly to A_2 subroot systems, each containing three (positive) roots, so that each line contains three of the 36 points. The 120 lines are determined as follows. Consider a triad of triples of azygetic double sixes and the corresponding matrix of linear forms (see (95)), say

$$[ijk.lmn] = \begin{bmatrix} h_{ij} & h_{jk} & h_{ik} \\ h_{lm} & h_{mn} & h_{ln} \\ h & h_{ijk} & h_{lmn} \end{bmatrix}.$$

Taking the ideal defined by the vanishing of two rows defines the corresponding line, i.e.,

$$\begin{aligned} L_{\{ij.jk\}} &= \langle h_{lm}, h_{mn}, h_{ln}, h, h_{ijk}, h_{lmn} \rangle \\ L_{\{lm.mn\}} &= \langle h_{ij}, h_{jk}, h_{ik}, h, h_{ijk}, h_{lmn} \rangle \\ L_{\{ijk\}} &= \langle h_{ij}, h_{jk}, h_{ik}, h_{lm}, h_{mn}, h_{ln} \rangle. \end{aligned} \tag{103}$$

⁴The arrangement is $\mathcal{A}(\mathbf{D}_4)$, minus the plane at infinity. Of the 16 $t_3(1)$ of $\mathcal{A}(\mathbf{D}_4)$ (see (9)), four lie in the plane at infinity.

⁵Likewise, nine of the 12 t_6 of $\mathcal{A}(\mathbf{D}_4)$ lie in the plane at infinity

Each of the 120 lines contains three of the nodes, so for example,

$$H_{ij}, H_{jk}, H_{ik} \in L_{\{ij,jk\}}. \quad (104)$$

There are 40 such triples of the 120 lines, which have the characterising property that they span \mathbb{P}^5 . These correspond to subroot systems of the type $A_2 \times A_2 \times A_2$, where all three copies are orthogonal to one another. Note that given an A_2 subroot system, there is a unique $A_2 \times A_2$ subroot system orthogonal to it. Thus the A_2 subroot system is defined by the vanishing of the six root forms of the complementary $A_2 \times A_2$. There are 120 of each of both types of subroot systems. Summing up, there are six of the 36 hyperplanes passing through each of these 120 lines while each such line contains three of the 36 nodes.

The *induced arrangement* is as follows. Blowing up along the line introduces an exceptional \mathbb{P}^3 over each point of the line; the intersection of it with the proper transforms of the six planes passing through it is the induced arrangement. It is of type $A_2 \times A_2$, i.e., is given by two skew lines in \mathbb{P}^3 and two sets of three hyperplanes through each line.

6.5.4 216 \mathbb{P}^1 's

Consider a pair of *skew* lines, say a_1, a_2 . There is a unique double six containing the given pair as a column, e.g.,

$$N_{12} = \begin{bmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{bmatrix}.$$

There are 216 lines in \mathbb{P}^5 which join points such as A_1, A_2, H_{12} (see (98) and (102)). The ideal of these 216 lines is generated by 24 sextics, forming the irreducible representation denoted 24_p in [BL]. We can exhibit these sextics explicitly, as follows. The 216 lines are given by the equations:

$$\begin{aligned} \langle A_i, A_j, H_{ij} \rangle &= \langle h_{kl}|k, l \neq i, j; h_{ijk} \rangle & (15) \\ \langle B_i, B_j, H_{ij} \rangle &= \langle h_{kl}|k, l \neq i, j; h_{klm}|k, l, m \neq i, j \rangle & (15) \\ \langle A_i, B_i, H \rangle &= \langle h_{kl}|k, l \neq i \rangle & (6) \\ \langle A_i, C_{jk}, H_{lmn} \rangle &= \langle h_{jk}, h_{\lambda\mu}|\lambda, \mu \neq i, j, k; h_{ij\lambda}, h_{ik\lambda}|\lambda \neq i, j, k \rangle & (60) \\ \langle B_i, C_{jk}, H_{ijk} \rangle &= \langle h_{jk}, h_{\lambda\mu}|\lambda, \mu \neq i, j, k; h_{\lambda\mu\nu}|\lambda \neq i, j, k, \mu \neq i, k, \nu \neq i \rangle & (60) \\ \langle C_{ik}, C_{jk}, H_{ij} \rangle &= \langle h_{ijm}, h_{mn}, h, h_{kmn}|m, n \neq i, j, k \rangle & (60), \end{aligned} \quad (105)$$

i.e., each is defined by the vanishing of ten of the h 's; these lines are the $t_{10}(1)$ listed in the table of the arrangement. We claim the sextics are the products of the six root forms of an $A_2 \times A_2$ subroot system. To see this, pick one, say $\Phi = h_{12} \cdot h_{13} \cdot h_{23} \cdot h_{45} \cdot h_{46} \cdot h_{56}$. It will suffice to check that for any of the 216 lines listed in (105), at least one of the hyperplanes on the right hand side is among the set $h_{12}, h_{13}, h_{23}, h_{45}, h_{46}, h_{56}$. This is at most a tedious, but straightforward task.

The dual \mathbb{P}^3 's, which are defined by the vanishing of the forms which are dual to the points of the left-hand sides, each *contain* the ten points which are dual to the forms on the right, for example

$$P_{\langle A_i, A_j \rangle} = \{a_i = a_j = h_{ij} = 0\} \ni (H_{kl}, H_{ijk}).$$

The induced arrangement over each line is the arrangement $\mathcal{A}(W(\mathbf{A}_4))$ of (9). Among the ten hyperplanes defining the line, say $\langle A_1, B_1, H \rangle$, there are ten triples of azygetic double sixes, $\{ij.jk\}$ in (94), with $i \neq 1$: $\{23.34\}$, $\{23.35\}$, $\{23.36\}$, $\{24.45\}$, $\{24.46\}$, $\{25.56\}$, $\{34.45\}$, $\{34.46\}$, $\{35.56\}$, and $\{45.56\}$, and these determine the ten $t_3(1)$ of (9).

6.5.5 45 \mathbb{P}^1 's

The 45 lines are the lines joining the 27 points of (102) in threes, for example,

$$L_{(ij)} = \langle A_i, B_j, C_{ij} \rangle.$$

These lines are defined by the vanishing of 12 of the h 's, so for example

$$L_{(12)} = \langle h_{34}, h_{35}, h_{36}, h_{45}, h_{46}, h_{56}, h_{234}, h_{235}, h_{236}, h_{245}, h_{246}, h_{256} \rangle; \quad (106)$$

these are the hyperplanes corresponding to the 12 double sixes *not* containing any of a_1, b_2, c_{12} .

The induced arrangement is $\mathcal{A}(W(\mathbf{D}_4))$, with 12 planes corresponding to the 12 hyperplanes through the line. Again there will be $t_2(1)$'s and $t_3(1)$'s, corresponding to triples of azygetic double sixes (respectively to syzygetic double sixes).

The ideal of the 45 lines is generated by 15 quartics which form the irreducible representation denoted 15_q in [BL]. It is easy to see that this space of quartics is generated by a product of four pairwise azygetic h 's, for example by $h_{24} \cdot h_{124} \cdot h_{35} \cdot h_{135}$. In fact, each hyperplane of type h_{ij} contains the 15 lines numbered (like the tritangents) by:

$$\begin{array}{ll} (kl) & \text{for } k, l \neq i, j \quad (12 \text{ of these}) \\ (ij.kl.mn) & \text{for } k, l, m, n \neq i, j \quad (3 \text{ of these}) \end{array}$$

while the hyperplanes of type h_{ijk} contain the 15 lines numbered by:

$$\begin{array}{ll} (mn) & \text{for } n = i \text{ or } j, m \neq i, j \quad (9 \text{ of these}) \\ (il.jm.kn) & \quad (6 \text{ of these}). \end{array}$$

It is now easy to check that every line is contained in at least one of the four hyperplanes. Alternatively we can argue as follows: each h contains 15 of the lines; there are six \mathbb{P}^3 's which are the intersection of two of the four, three of which are contained in each h . These three meet in a common line in the h , so the number of lines contained in the union is: $4 \cdot (15 - 7) + 2 \cdot 6 + 1 = 45$, where the 7=number of lines in the union of the three \mathbb{P}^2 's in each h , 2=3-1 is the number of lines in each such \mathbb{P}^2 , not in the others, and one is the common line. Note that this is the Macdonald representation corresponding to the four roots of an $A_1 \times A_1 \times A_1 \times A_1 \subset D_4$ subroot system. Five of these lines meet at each of the 27 points, corresponding to the five tritangents through each of the 27 lines.

6.5.6 36 points

These are the 36 points (98) dual to the 36 hyperplanes of 6.5.1. The induced arrangement is of course just the arrangement in \mathbb{P}^4 above. There are 15 hyperplanes passing through each of the 36 points, and these are just the hyperplanes which are coded by the double sixes which are syzygetic to the one with the notation of the point as in (98). So, for example, the 15 \mathbb{P}^4 's through the point H are the 15 h_{ij} .

These points correspond to (\pm) the roots of E_6 . The orthogonal complement in \mathbb{R}^6 of the root α is projectively equivalent to the *dual* hyperplane to the point. For example, H is dual to h , and one of the hyperplanes P will contain $H \iff$ the dual point p is contained in h . The ideal of the 36 points is generated by 20 cubics, forming the irreducible representation of $W(E_6)$ denoted 20_p in [BL]. We can find these cubics explicitly as follows. Consider the hyperplanes a_1, b_2, c_{12} corresponding to a tritangent. From Table 5 above we see that each of these hyperplanes contains 20 of the 36 points (actually, the table contains the dual information: there are 20 of the h_{ij} , etc, passing through each of the 27 points), and the \mathbb{P}^3 which is the common intersection of these three contains 12 of the 36 (the dual information is contained in the table: the 45 lines are 12-fold lines). Hence the product $a_1 \cdot b_2 \cdot c_{12}$ contains $3 \cdot (20-12) + 12 = 36$ of the 36 points.

Through each of the 36 points, also 15 of the 27 hyperplanes of (99) pass, corresponding to the 15 lines *not* contained in the double six whose notation the point has. For example, the point $H = \frac{1}{2}(1, \dots, 1, 3)$ is contained in all the c_{ij} . In the exceptional \mathbb{P}^4 at the point, both sets of 15 hyperplanes (coming from the 36, respectively 27 hyperplanes) *coincide*.

6.5.7 27 points

These are the points A_i, B_i, C_{ij} of (102). There are 20 of the 36 hyperplanes meeting at each, so the induced arrangement is one of 20 \mathbb{P}^3 's in \mathbb{P}^4 , and one sees easily that it is $\mathcal{A}(W(\mathbf{D}_5))$. This arrangement is also induced in any of the hyperplanes a_i, b_i, c_{ij} of (99); we note that there are two kinds of \mathbb{P}^2 , namely $t_2(2)$'s, corresponding to pairs of skew lines, and $t_3(2)$'s, corresponding to tritangents. Since each line is contained in five tritangents, there are five of the latter and 15 of the former (in each a_i , etc.). These 15 form an arrangement of type $\mathcal{A}(W(\mathbf{A}_5))$ as discussed above. The ideal of these 27 points is generated by 30 cubics, forming the irreducible representation denoted 30_p in [BL]. It is easy to see that this space of cubics is generated by a product of three

members of a syzgetic triple as $h_{12} \cdot h_{13} \cdot h_{23}$, for example: Each of the hyperplanes contains 15 of the 27, the \mathbb{P}^3 which is their common intersection contains nine, so the union contains $3 \cdot (15-9) + 9 = 27$, or all of the points. Note that this is just the Macdonald representation corresponding to the (3) roots of an A_2 subroot system.

We need, in addition to the above, certain information on the dual spaces.

6.5.8 45 \mathbb{P}^3 's

Consider one of the 45 \mathbb{P}^3 's which is dual to one of the 45 lines of 6.5.5; it is cut out by three of the forms (99), and can be denoted as one of the 45 tritangents, for example, if $L_{(ij)}$ denotes the line $\langle A_i, B_j, C_{ij} \rangle$ as in (106), the dual \mathbb{P}^3 may be denoted by $l_{(ij)}$, and

$$\begin{aligned} l_{(ij)} &= a_i \cap b_j \cap c_{ij} \\ l_{(ij:kl:mn)} &= c_{ij} \cap c_{kl} \cap c_{mn}. \end{aligned} \quad (107)$$

Consider the \mathbb{P}^3 $l_{(12)}$, given by $a_1 = b_2 = c_{12} = 0$, or $x_1 = x_6 = 0$. Then one checks easily that the hyperplanes (96) reduce in $l_{(12)}$ to the arrangement $\mathcal{A}(\mathbf{F}_4)$ of (7). Considering how the 27 hyperplanes (99) intersect $l_{(12)}$, we find that these restrict to the set of short roots, that is, give a subarrangement of type $\mathcal{A}(\mathbf{D}_4)$. See also Proposition 8.1 below.

6.6 Invariants

Since the 27 forms (99) are (as a set) invariant under the Weyl group the expression

$$I_k := \sum_{i,j} \{a_i^k + b_i^k + c_{ij}^k\}, \quad (108)$$

if non-vanishing, is an invariant of degree k . The ring of invariants of $W(E_6)$ is generated by elements in degrees 2, 5, 6, 8, 9 and 12, which can be taken to be I_2, \dots, I_{12} . We note that while I_2 and I_5 are *unique*, the other invariants are only defined up to addition of terms coming from lower degrees.

7 The invariant quintic

7.1 Equation

There is a unique (up to scalars) $W(E_6)$ -invariant polynomial of degree 5. Written with integer coefficients in the variables x_i it is

$$f(x_1, \dots, x_6) = x_6^5 - 6x_6^3\sigma_1(x) - 27x_6(\sigma_1^2(x) - 4\sigma_2(x)) - 648\sqrt{\sigma_5(x)}, \quad (109)$$

where $\sigma_i(x)$ is the i th elementary symmetric polynomial of the x_1^2, \dots, x_5^2 , so in particular $\sqrt{\sigma_5(x)} = x_1x_2x_3x_4x_5$. The polynomial $f(x)$ displays manifestly the $W(D_5)$ -invariance of the quintic. Under the change of variables from the x_i to the a_i of (99), the equation $g(a)$ can be derived as follows. By (101), we have $b_i = a_i - \frac{1}{3}(a_1 + \dots + a_6)$, which by equation (100) can be written $b_i = a_i - h$. The following trick was shown to me by I. Naruki. Consider $\prod_{i=1}^6 a_i - \prod_{i=1}^6 b_i$. This sextic divides the root h , and the quotient is $W(E_6)$ -invariant. To see this, calculate

$$\begin{aligned} a_1 \cdots a_6 - (b_1 \cdots b_6) &= \prod a_i - \prod (a_i - h) \\ &= \sigma_6(a) - [\sigma_6(a) - h\sigma_5(a) + h^2\sigma_4(a) - h^3\sigma_3(a) + h^4\sigma_2(a) - h^5\sigma_1(a)], \end{aligned} \quad (110)$$

where here $\sigma_i(a)$ are the elementary symmetric functions of the a_i . Consequently,

$$a_1 \cdots a_6 - (b_1 \cdots b_6) = h(\sigma_5(a) - h\sigma_4(a) + h^2\sigma_3(a) - h^3\sigma_2(a) + h^4\sigma_1(a)),$$

and since by (100) $h = -\frac{1}{3}\sigma_1(a)$, this yields

$$g(a) = 81\sigma_5(a) + 27\sigma_4(a)\sigma_1(a) + 9\sigma_3(a)\sigma_1^2(a) + 3\sigma_2(a)\sigma_1^3(a) + \sigma_1^5(a), \quad (111)$$

giving the expression of the invariant quintic expressing manifestly the $W(A_5) = \Sigma_6$ -invariance.

Definition 7.1 The *invariant quintic* \mathcal{I}_5 is the hypersurface of degree 5

$$\mathcal{I}_5 := \{x \in \mathbb{P}^5 \mid f(x) = 0\} \cong \{a \in \mathbb{P}^5 \mid g(a) = 0\},$$

where the isomorphism is given by the change of coordinates from the x_i to the a_i .

7.2 Singular locus

Because of the equivalence of the coordinates x_i , $i = 1, \dots, 5$, there are essentially two different partial derivatives of f , namely

$$\begin{aligned} j_1 &:= \frac{\partial f}{\partial x_1} \cong \dots \cong \frac{\partial f}{\partial x_5} \\ j_2 &:= \frac{\partial f}{\partial x_6}. \end{aligned} \tag{112}$$

Calculating these forms gives

$$\begin{aligned} -\frac{\partial f}{\partial x_i} &= 12x_6^3x_i + 54x_6x_i(x_j^2 + x_k^2 + x_l^2 + x_m^2) + 648x_jx_kx_lx_m, \\ \frac{\partial f}{\partial x_6} &= 5x_6^4 - 18x_6^2\sigma_1(x) + 27(\sigma_1^2(x) - 4\sigma_4(x)). \end{aligned} \tag{113}$$

These are quartics with manifest $W(D_4)$ and $W(D_5)$ symmetry, respectively.

Theorem 7.2 *The singular locus of \mathcal{I}_5 consists of the 120 lines of 6.5.3, which meet ten at a time in the 36 points of 6.5.6.*

Proof: Consider first the hyperplane section $x_6 = 0$. Then the equations to be solved are

$$\begin{aligned} x_ix_jx_kx_l &= 0, \quad (i, j, k, l < 6); \\ \sigma_1^2(x) - 4\sigma_2(x) &= 0. \end{aligned} \tag{114}$$

From (114) we get: two of the x_i must vanish, say x_4, x_5 , and then (115) takes the form

$$\begin{aligned} (x_1^2 + x_2^2 + x_3^2)^2 - 4(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) &= 0, \\ (x_1 + x_2 + x_3)(x_1 - x_2 - x_3)(x_1 + x_2 - x_3)(x_1 - x_2 + x_3) &= 0 \end{aligned} \tag{116}$$

which splits into a product of four lines. Since the product $x_1 \cdots x_5 = 0$ is a coordinate simplex in $\mathbb{P}^4 = \{x_6 = 0\}$, it follows that the 2-simplices of this simplex correspond to planes where two of the coordinates vanish, hence there are $\binom{5}{2} = 10$ such 2-simplices; in each we have the four lines given by (116). This gives the 40 of the 120 lines contained in $x_6 = 0$. This implies that, in the union of the 27 hyperplanes (99), the singular locus of \mathcal{I}_5 consists of 120 lines.

Suppose that $x_6 \neq 0$. Then the simultaneous vanishing of the partials $\frac{\partial f}{\partial x_i}$, $i = 1, \dots, 5$ imply that four of the x_i must vanish, say $x_2 = x_3 = x_4 = x_5 = 0$. But the intersection of \mathcal{I}_5 with the line $\{x_2 = x_3 = x_4 = x_5 = 0\}$ is given by

$$x_6^5 - 6x_6^3x_1^2 - 27x_6x_1^4 = x_6(x_6 + i\sqrt{3}x_1)(x_6 - i\sqrt{3}x_1)(x_6 + 3x_1)(x_6 - 3x_1), \tag{117}$$

and the last two terms are the equations of b_2 and c_{12} , two other of the 27 hyperplanes of (99). From this we conclude that any singular point is contained in one of the 27 hyperplanes, and by the above, that the singular locus of \mathcal{I}_5 consists of the 120 lines, as stated. \square

The types of singularities are given by the following.

Proposition 7.3 *The singularities of \mathcal{I}_5 can be described as follows.*

i) At a generic point $x \in$ one of the 120 lines, a transversal hyperplane section has an ordinary A_1 -singularity, so the singularity is of type $disc \times A_1$.

ii) At one of the 36 intersection points p , the singularity has multiplicity 3, and the tangent cone is of the form

$$s_5 + s_4 t + s_3 t^2,$$

where s_3 is the cone over the Segre cubic, $s_4 = s_3 \cdot h_p$, where h_p is the hyperplane of 6.5.1 dual to p , and s_5 is the cone over the intersection $\mathcal{I}_5 \cap h_p$.

Proof: i) follows from consideration of generic hyperplane sections of \mathcal{I}_5 , which are quintic threefolds with 120 isolated singularities, so singularities worse than $disc \times A_1$ are impossible. ii) is just a computation, done as follows. Suppose the point is $p = H_{23} = (1, -1, 0, 0, 0)$. Then inhomogenizing by setting $t_i = x_i/x_1 - tp_i$ (where p_i denotes the i th coordinate of p), inserting into the equation of \mathcal{I}_5 gives the stated result. The fact that s_3 is the cone over \mathcal{S}_3 can be seen as follows. We can write

$$f = s_5 + s_3(th_p + t^2) \tag{118}$$

and it follows that blowing up \mathcal{I}_5 at p is given by setting $t = \infty$ and that the proper transform of \mathcal{I}_5 in the exceptional \mathbb{P}^4 (of the blow up of \mathbb{P}^5 at the point p) is a cubic $\mathcal{S}_3 = 0$, where $s_3 = 0$ is the cone over $\mathcal{S}_3 = 0$. Since there are ten of the 120 lines meeting at p , the resolving divisor of the blow up, which is a cubic threefold, will have ten isolated singularities. As mentioned already above, this implies the cubic threefold is isomorphic to \mathcal{S}_3 . One can also see the 15 special hyperplane sections of \mathcal{S}_3 : these are the proper transforms, under the blowing up of p , of the 15 of the 36 hyperplanes 6.5.1 passing through the point. The rest is calculation. \square

We have (using Macaulay) calculated the ideal $\mathcal{J}(120)$ of the 120 lines, and it turns out to be just the Jacobian ideal of \mathcal{I}_5 . I know of no simple proof of this fact.

7.3 Resolution of singularities

It turns out to be very easy to desingularize \mathcal{I}_5 . By the proof of Proposition 7.3, we know the 36 triple points can be resolved by blowing up each such point p . Let $\rho^{(1)} : \mathcal{I}_5^{(1)} \rightarrow \mathcal{I}_5$ denote this blow up of \mathcal{I}_5 . This has the effect of separating all 120 lines of \mathcal{I}_5 , and the singularities along the lines are just A_1 , by Proposition 7.3, i). Hence a desingularisation is achieved by resolving each of the 120 lines. There are two possible ways to do this. First, one can blow up the lines in $\mathbb{P}^{5(1)} = \mathbb{P}^5$ blown up in the 36 points, and take the proper transform of $\mathcal{I}_5^{(1)}$; this has the effect of replacing each singular line by a quadric surface bundle, a $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle, over the line. Hence there are 120 exceptional divisors, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We call this resolution of \mathcal{I}_5 the *big resolution* and denote it by $\tilde{\mathcal{I}}_5$. Secondly, we can take a small resolution by blowing down one of the fiberings in the exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ over a point of the line. In this way, each of the singular lines is replaced by a \mathbb{P}^1 -bundle over the line, in other words by a $\mathbb{P}^1 \times \mathbb{P}^1$. We call this the *small resolution* and denote it by $\mathcal{I}_5^{(s)}$. Here no further (beyond the 36 on $\mathcal{I}_5^{(1)}$) exceptional divisors are introduced.

Lemma 7.4 *The quintic \mathcal{I}_5 has two resolutions, which we denote by $\tilde{\mathcal{I}}_5$ and $\mathcal{I}_5^{(s)}$. On $\tilde{\mathcal{I}}_5$ there are 36+120 exceptional divisors, 36 copies of the resolution of the Segre cubic, and 120 copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. On $\mathcal{I}_5^{(s)}$ there are only 36 exceptional divisors, each a small resolution of the Segre cubic.*

7.4 \mathcal{I}_5 is rational

Quite generally, in \mathbb{P}^5 , taking four \mathbb{P}^3 's which meet only in lines, there is a unique line which meets each of them and passes through a given point $P \in \mathbb{P}^5$, namely the line $\langle \alpha, P \rangle \cap \langle \beta, P \rangle \cap \langle \gamma, P \rangle \cap \langle \delta, P \rangle$, if $\alpha, \beta, \gamma, \delta$ denote the \mathbb{P}^3 's and $\langle \alpha, P \rangle$ is the hyperplane spanned by α and P . Now let $P \in \mathcal{I}_5$, and choose four of the 15 of the 45 \mathbb{P}^3 's through one of the triple points p , such that the four \mathbb{P}^2 's on $(\mathcal{S}_3)_p$ meet each other only in points; then the four \mathbb{P}^3 's meet only in lines, and we may apply this reasoning to conclude:

for each $P \in \mathcal{I}_5 - \{4 \mathbb{P}^3\text{'s}\}$, there is a unique line L_p , which joins P and $\alpha, \beta, \gamma, \delta$.

Then, fixing a generic hyperplane $F \subset \mathbb{P}^5$. the line L_p intersects F in a single point; we get a rational map:

$$\mathcal{I}_5 \dashrightarrow F$$

$$P \mapsto L_p \cap F.$$

We now carry out this argument to derive an explicit rationalisation. I am indebted to B. v. Geemen for help in performing this. We *choose* four convenient \mathbb{P}^3 's which only meet in lines (although these do not all pass through a point). The four \mathbb{P}^3 's will be defined as follows:

$$\begin{aligned} P_1 &= \{a_1 = b_4 = 0\} = \{l_1 = m_1 = 0\} \\ P_2 &= \{a_4 = b_5 = 0\} = \{l_2 = m_2 = 0\} \\ P_3 &= \{a_5 = b_6 = 0\} = \{l_3 = m_3 = 0\} \\ P_4 &= \{c_{35} = c_{12} = 0\} = \{l_4 = m_4 = 0\} \end{aligned} \quad (119)$$

Letting F be an auxilliary \mathbb{P}^4 with homogenous coordinates $(y_0 : \dots : y_4)$, the intersection of the line $\langle P_1, \alpha \rangle \cap \dots \cap \langle P_4, \alpha \rangle$ with F is given by

$$y_0 l_i - y_i m_i = 0,$$

which leads to

$$\begin{aligned} y_0 &= m_1 \cdots m_4 \\ y_i &= l_i \cdot m_1 \cdots \widehat{m}_i \cdots m_4, \end{aligned} \quad (120)$$

a system of quartics in \mathbb{P}^5 , which, when restricted to \mathcal{I}_5 , give the rational map $\varphi : \mathcal{I}_5 \dashrightarrow \mathbb{P}^4 (= F)$. Inverting the equations for x_1, \dots, x_6 we get

$$\begin{aligned} x_1 &= y_0^6 y_1 y_3 + y_0^5 y_1^2 y_3 - 2y_0^6 y_2 y_3 - y_0^5 y_1 y_2 y_3 + y_0^4 y_1^2 y_2 y_3 + 2y_0^4 y_1 y_2 y_3^2 + 2y_0^3 y_1^2 y_2 y_3^2 \\ &\quad - y_0^6 y_1 y_4 - y_0^5 y_1^2 y_4 - y_0^5 y_1 y_2 y_4 - 2y_0^4 y_1^2 y_2 y_4 - y_0^3 y_1^2 y_2^2 y_4 - y_0^5 y_1 y_3 y_4 - y_0^4 y_1^2 y_3 y_4 \\ &\quad - 2y_0^4 y_1 y_2 y_3 y_4 - 4y_0^3 y_1^2 y_2 y_3 y_4 - 2y_0^3 y_1 y_2^2 y_3 y_4 - 3y_0^2 y_1^2 y_2^2 y_3 y_4 - 2y_0^3 y_1 y_2 y_3^2 y_4 \\ &\quad - 3y_0^2 y_1^2 y_2 y_3^2 y_4 - 2y_0^2 y_1 y_2^2 y_3^2 y_4 - 2y_0 y_1^2 y_2^2 y_3^2 y_4 + y_0^2 y_1^2 y_2 y_3 y_4^2 \\ &\quad + y_0 y_1^2 y_2^2 y_3 y_4^2 + y_0 y_1^2 y_2 y_3 y_4^2 + y_1^2 y_2^2 y_3^2 y_4^2 \\ x_2 &= y_0^6 y_1 y_3 + y_0^5 y_1^2 y_3 + y_0^5 y_1 y_2 y_3 + y_0^4 y_1^2 y_2 y_3 + 2y_0^5 y_1 y_3^2 + 2y_0^4 y_1^2 y_3^2 + 2y_0^4 y_1 y_2 y_3^2 \\ &\quad + 2y_0^3 y_1^2 y_2 y_3^2 - y_0^6 y_1 y_4 - y_0^5 y_1^2 y_4 - y_0^5 y_1 y_2 y_4 - 2y_0^4 y_1^2 y_2 y_4 - y_0^3 y_1^2 y_2^2 y_4 - 5y_0^5 y_1 y_3 y_4 \\ &\quad - 5y_0^4 y_1^2 y_3 y_4 - 6y_0^4 y_1 y_2 y_3 y_4 - 8y_0^3 y_1^2 y_2 y_3 y_4 - 2y_0^3 y_1 y_2^2 y_3 y_4 - 3y_0^2 y_1^2 y_2^2 y_3 y_4 - 4y_0^4 y_1 y_3^2 y_4 \\ &\quad - 4y_0^3 y_1^2 y_3^2 y_4 - 6y_0^3 y_1 y_2 y_3^2 y_4 - 7y_0^2 y_1^2 y_2 y_3^2 y_4 - 2y_0^2 y_1 y_2^2 y_3^2 y_4 - 2y_0 y_1^2 y_2^2 y_3^2 y_4 + 2y_0^5 y_1 y_4^2 \\ &\quad + 2y_0^4 y_1^2 y_4^2 + 2y_0^4 y_1 y_2 y_4^2 + 4y_0^3 y_1^2 y_2 y_4^2 + 2y_0^2 y_1^2 y_2^2 y_4^2 + 4y_0^4 y_1 y_3 y_4^2 + 4y_0^3 y_1^2 y_3 y_4^2 \\ &\quad + 6y_0^3 y_1 y_2 y_3 y_4^2 + 9y_0^2 y_1^2 y_2 y_3 y_4^2 + 2y_0^2 y_1 y_2^2 y_3 y_4^2 + 5y_0 y_1^2 y_2^2 y_3 y_4^2 + 2y_0^3 y_1 y_3^2 y_4^2 + 2y_0^2 y_1^2 y_3^2 y_4^2 \\ &\quad + 4y_0^2 y_1 y_2 y_3^2 y_4^2 + 5y_0 y_1^2 y_2 y_3^2 y_4^2 + 2y_0 y_1 y_2^2 y_3^2 y_4^2 + 3y_1^2 y_2^2 y_3^2 y_4^2 \\ x_3 &= 2y_0^7 y_3 + 3y_0^6 y_1 y_3 + y_0^5 y_1^2 y_3 + 2y_0^6 y_2 y_3 + 3y_0^5 y_1 y_2 y_3 + y_0^4 y_1^2 y_2 y_3 - 2y_0^7 y_4 - 3y_0^6 y_1 y_4 \\ &\quad - y_0^5 y_1^2 y_4 - 2y_0^6 y_2 y_4 - 5y_0^5 y_1 y_2 y_4 - 2y_0^4 y_1^2 y_2 y_4 - 2y_0^4 y_1 y_2^2 y_4 - y_0^3 y_1^2 y_2^2 y_4 - 2y_0^6 y_3 y_4 \\ &\quad - 3y_0^5 y_1 y_3 y_4 - y_0^4 y_1^2 y_3 y_4 - 4y_0^5 y_2 y_3 y_4 - 6y_0^4 y_1 y_2 y_3 y_4 - 2y_0^3 y_1^2 y_2 y_3 y_4 - 2y_0^3 y_1 y_2^2 y_3 y_4 \\ &\quad - y_0^2 y_1^2 y_2^2 y_3 y_4 + 2y_0^3 y_1 y_2 y_3^2 y_4 + y_0^2 y_1^2 y_2 y_3^2 y_4 - 2y_0^3 y_1 y_2 y_3 y_4^2 \\ &\quad - y_0^2 y_1^2 y_2 y_3 y_4^2 - 2y_0^2 y_1 y_2^2 y_3 y_4^2 - y_0 y_1^2 y_2^2 y_3 y_4^2 - 2y_0^2 y_1 y_2 y_3^2 y_4^2 - y_0 y_1^2 y_2 y_3^2 y_4^2 \\ &\quad - 2y_0 y_1 y_2^2 y_3^2 y_4^2 - y_1^2 y_2^2 y_3^2 y_4^2 \\ x_4 &= 2y_0^7 y_3 + 3y_0^6 y_1 y_3 + y_0^5 y_1^2 y_3 + y_0^5 y_1 y_2 y_3 + y_0^4 y_1^2 y_2 y_3 - 2y_0^5 y_1 y_3^2 - 2y_0^4 y_1^2 y_3^2 - 2y_0^7 y_4 \\ &\quad - 3y_0^6 y_1 y_4 - y_0^5 y_1^2 y_4 - 3y_0^5 y_1 y_2 y_4 - 2y_0^4 y_1^2 y_2 y_4 - y_0^3 y_1^2 y_2^2 y_4 - 2y_0^6 y_3 y_4 - y_0^5 y_1 y_3 y_4 \\ &\quad + y_0^4 y_1^2 y_3 y_4 - 2y_0^4 y_1 y_2 y_3 y_4 - y_0^2 y_1^2 y_2^2 y_3 y_4 + 4y_0^4 y_1 y_3^2 y_4 + 4y_0^3 y_1^2 y_3^2 y_4 + 2y_0^3 y_1 y_2 y_3^2 y_4 \\ &\quad + 3y_0^2 y_1^2 y_2 y_3^2 y_4 - 2y_0^4 y_1 y_3 y_4^2 - 2y_0^3 y_1^2 y_3 y_4^2 - 2y_0^3 y_1 y_2 y_3 y_4^2 - 3y_0^2 y_1^2 y_2 y_3 y_4^2 \\ &\quad - y_0 y_1^2 y_2^2 y_3 y_4^2 - 2y_0^3 y_1 y_3^2 y_4^2 - 2y_0^2 y_1^2 y_3^2 y_4^2 - 2y_0^2 y_1 y_2 y_3^2 y_4^2 - 3y_0 y_1^2 y_2 y_3^2 y_4^2 - y_1^2 y_2^2 y_3^2 y_4^2 \\ x_5 &= -y_0^6 y_1 y_3 - y_0^5 y_1^2 y_3 - y_0^5 y_1 y_2 y_3 - y_0^4 y_1^2 y_2 y_3 + y_0^6 y_1 y_4 + y_0^5 y_1^2 y_4 + 2y_0^6 y_2 y_4 + 3y_0^5 y_1 y_2 y_4 \\ &\quad + 2y_0^4 y_1^2 y_2 y_4 + 2y_0^4 y_1 y_2^2 y_4 + y_0^3 y_1^2 y_2^2 y_4 + 3y_0^5 y_1 y_3 y_4 + 3y_0^4 y_1^2 y_3 y_4 + 2y_0^4 y_1 y_2 y_3 y_4 \\ &\quad + 4y_0^3 y_1^2 y_2 y_3 y_4 + 2y_0^3 y_1 y_2^2 y_3 y_4 + y_0^2 y_1^2 y_2^2 y_3 y_4 + y_0^2 y_1 y_2 y_3^2 y_4 - 2y_0^5 y_1 y_4^2 - 2y_0^4 y_1^2 y_4^2 \\ &\quad - 2y_0^4 y_1 y_2 y_4^2 - 4y_0^3 y_1^2 y_2 y_4^2 - 2y_0^2 y_1^2 y_2^2 y_4^2 - 2y_0^4 y_1 y_3 y_4^2 - 2y_0^3 y_1^2 y_3 y_4^2 - 2y_0^3 y_1 y_2 y_3 y_4^2 \\ &\quad - 5y_0^2 y_1^2 y_2 y_3 y_4^2 - 3y_0 y_1^2 y_2 y_3 y_4^2 - y_0 y_1^2 y_2 y_3^2 y_4^2 - y_1^2 y_2^2 y_3^2 y_4^2 \\ x_6 &= -3y_0^6 y_1 y_3 - 3y_0^5 y_1^2 y_3 - 3y_0^5 y_1 y_2 y_3 - 3y_0^4 y_1^2 y_2 y_3 + 3y_0^6 y_1 y_4 + 3y_0^5 y_1^2 y_4 + 3y_0^5 y_1 y_2 y_4 \\ &\quad + 6y_0^4 y_1^2 y_2 y_4 + 3y_0^3 y_1^2 y_2^2 y_4 + 3y_0^5 y_1 y_3 y_4 + 3y_0^4 y_1^2 y_3 y_4 + 6y_0^4 y_1 y_2 y_3 y_4 + 6y_0^3 y_1^2 y_2 y_3 y_4 \\ &\quad + 3y_0^2 y_1^2 y_2^2 y_3 y_4 - 3y_0^2 y_1^2 y_2 y_3^2 y_4 + 3y_0^2 y_1^2 y_2 y_3 y_4^2 + 3y_0 y_1^2 y_2^2 y_3 y_4^2 + 3y_0 y_1^2 y_2 y_3^2 y_4^2 \\ &\quad + 3y_1^2 y_2^2 y_3^2 y_4^2 \end{aligned}$$

a system of octics in \mathbb{P}^4 , yielding the rational map

$$\psi : \mathbb{P}^4 \longrightarrow \mathcal{I}_5.$$

These rational maps are morphisms outside of the base locus.

Lemma 7.5 *The base locus of φ consists of the four \mathbb{P}^3 's P_1, P_2, P_3, P_4 . The base locus of ψ is a surface of degree 32. \square*

The first statement is clear from construction, while the second is a computation. We performed this with the help of Macaulay to calculate a standard basis of the ideal; the base locus is the intersection of the six octics.

8 Hyperplane sections

8.1 Reducible hyperplane sections

Consider the hyperplane section $H_5 := \mathcal{I}_5 \cap \{x_6 = 0\}$; it is the union of five \mathbb{P}^3 's which form a coordinate simplex $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5$ in the \mathbb{P}^4 given by $\{x_6 = 0\}$. Now $a_1 = -2/3x_6$ and invariance implies that the 27 hyperplane sections $a_i = 0, b_i = 0, c_{ij} = 0$ all have the same property. Each such hyperplane contains 40 of the 120 \mathbb{P}^1 's, which meet six at a time in 20 of the 36 points. Consider three lines in a tritangent, say (a_1, b_2, c_{12}) . These three hyperplanes pass through a common \mathbb{P}^3 , namely

$$l_{(12)} := \{x_6 = 0, x_1 = 0\}.$$

Such \mathbb{P}^3 's therefore correspond to the tritangents and there are 45 such on \mathcal{I}_5 ; these are just the 45 \mathbb{P}^3 's of 6.5.8. Hence we have

Proposition 8.1 *The quintic \mathcal{I}_5 contains 45 \mathbb{P}^3 's, which are cut out by the 27 hyperplane sections (99), and each such hyperplane section meets \mathcal{I}_5 in the union of five of the 45 \mathbb{P}^3 's. These can be numbered in terms of the tritangents of a cubic surface, i.e., for any 3 lines in a tritangent plane of a cubic surface, the corresponding hyperplanes of (99) intersect in a common \mathbb{P}^3 , and this \mathbb{P}^3 lies on \mathcal{I}_5 .*

Also the intersections of the 45 \mathbb{P}^3 's can be described. Each such \mathbb{P}^3 contains 16 of the 120 lines which meet in 12 of the 36 points; these 12 points are the vertices of a triad of desmic tetrahedra. Consider the $\mathbb{P}^3 l_{(12)}$; the corresponding tritangent meets 12 others, namely (13), (14), (15), (16), (32), (42), (52), (62), (12.34.56), (12.35.46), (12.36.45) and (21), and the 12 \mathbb{P}^3 's corresponding to them meet $l_{(12)}$ in a \mathbb{P}^2 (the generic intersection has dimension 1). These 12 planes in $l_{(12)}$ form the arrangement $\mathcal{A}(\mathbf{D}_4)$ in \mathbb{P}^3 .

8.2 Special hyperplane sections

We now consider the intersections of \mathcal{I}_5 with the 36 reflection hyperplanes 6.5.1. Take for example the reflection hyperplane $\{h = 0\}$; since h is just a multiple of $\sigma_1(a)$ (see (100)), it follows from the equation (111) that the intersection $\{h = 0\} \cap \mathcal{I}_5$ is a quintic hypersurface in \mathbb{P}^4 with the equation:

$$Q_1 := \{h = 0\} \cap \mathcal{I}_5 = \begin{cases} \sigma_1(a) = 0 \\ \sigma_5(a) = 0 \end{cases} \quad (122)$$

Comparing with the equation (69), we see that this is a copy of the Nieto quintic! By symmetry, each of the 36 hyperplane sections is isomorphic to this one, and we denote them by

$$T = \{h = 0\} \cap \mathcal{I}_5, T_{ij} = \{h_{ij} = 0\} \cap \mathcal{I}_5, T_{ijk} = \{h_{ijk} = 0\} \cap \mathcal{I}_5. \quad (123)$$

So we have:

Proposition 8.2 *There are 36 copies of the Nieto quintic (69) on \mathcal{I}_5 .*

We can determine the singular locus of these hyperplane sections, independently of the discussion given in section 4.1. The reflection hyperplane contains 20 of the 120 lines, which meet in 15 of the 36 points (corresponding to the 15 roots of an \mathbf{A}_5 subsystem), so the quintic threefold has 20 singular lines, with 15 singular points of multiplicity 3. In fact, the resolving divisor of each of these 15 points is a four-nodal cubic surface, which is a hyperplane section of the Segre cubic \mathcal{S}_3 (see the discussion following Problem 4.19, ii)). Furthermore, recalling that there are ten of the 120 lines which pass through the triple point which is *dual* to the given reflection hyperplane, each such intersects the reflection hyperplane transversally, giving the ten isolated ordinary double points on that quintic (see Proposition 4.1), and in some sense “explains” these isolated singularities.

8.3 Generic hyperplane sections

A generic hyperplane section is a quintic threefold in \mathbb{P}^4 with 120 nodes. This is a fascinating family of Calabi-Yau threefolds, which has a beautiful geometric configuration associated with it, in some sense “dual” to the configuration of the 27 lines on a cubic surface.

Proposition 8.3 *Let $H \in \mathbb{P}^5$ be a generic hyperplane and let $Q_H = \mathcal{I}_5 \cap H$ be the hyperplane section. Then we have*

- 1) *There are 45 \mathbb{P}^2 's on Q_H , which are cut out by 27 hyperplanes; these could appropriately be called quintangent planes.*
- 2) *The group of incidence preserving permutations of the 45 \mathbb{P}^2 's is $W(E_6)$; this is also the group of incidence preserving permutations of the 27 hyperplanes.*
- 3) *There are 36 hyperplane sections of Q_H , each of which is a 20-nodal quintic surface.*
- 4) *The 120 nodes of Q_H form an orbit under $W(E_6)$.*

Proof: For any of the 45 \mathbb{P}^3 's in \mathcal{I}_5 and hyperplane section H , it holds that $\mathbb{P}^3 \cap H = \mathbb{P}^2 \subset H \cap \mathcal{I}_5 = Q_H$, showing 1). The second point is evident, and in a sense “dual” to the situation with cubic surfaces. We have seen that a special hyperplane section as in (123) is isomorphic to the Nieto quintic and has 20 singular lines in its singular locus; therefore any generic hyperplane section has exactly 20 nodes. 4) follows since the $W(E_6)$ orbit consisting of the 120 lines, restricted to the hyperplane section is still an orbit. \square

We now consider some of the invariants of the nodal quintic threefolds. Let V denote a nodal quintic, $\widehat{V} \rightarrow V$ a small resolution and $\widetilde{V} \rightarrow V$ a big resolution. Letting s denote the number of nodes, the betti numbers are

$$\begin{aligned} b_1(V) &= 1 = b_1(\widehat{V}), & b_2(\widetilde{V}) &= 1 + d + s; \\ b_2(V) &= 1 + d = b_2(\widehat{V}), & b_4(\widetilde{V}) &= 1 + d + s; \\ b_3(V) &= b_3(V_t) - s + d, & b_3(\widehat{V}) &= b_3(V) - s + d = b_3(\widetilde{V}), \end{aligned} \tag{124}$$

where V_t is a smooth hypersurface of same degree as V and d is the *defect*. The defect may be calculated by the following result.

Theorem 8.4 ([W], p. 27) *Let $V \in \mathbb{P}^4$ be a nodal hypersurface of degree $n \geq 3$. Then*

$$\dim(\mathcal{P}_{2n-5}(V)) = \dim \left\{ \begin{array}{l} \text{homogenous polynomials of degree } 2n- \\ 5 \text{ in } \mathbb{P}^4, \text{ containing all nodes of } V \end{array} \right\} = \binom{2n-1}{4} - s + d.$$

Applied to the case at hand, we need the dimension of the space of *quintics* vanishing at all the nodes. Clearly this is the degree five component in the ideal of the 120 points. As we mentioned above, we *know* the ideal of the 120 lines (it is the Jacobian ideal of \mathcal{I}_5 $\mathcal{J}ac(\mathcal{I}_5)$), so we know also the ideal of the 120 points; it is the restriction of $\mathcal{J}ac(\mathcal{I}_5)$ to the hyperplane, generated by six quartics.

Proposition 8.5 *The dimension of the space $\mathcal{P}_5(Q_H)$ is 30.*

Proof: Each of the quartics (which are clearly independent for a generic hyperplane H) of $\mathcal{J}ac(\mathcal{I}_5)$ can be multiplied by any hyperplane, giving a quintic which contains the 120 nodes. The set of hyperplanes is $(\mathbb{P}^5)^\vee$, so the dimension of $\mathcal{P}_5(Q_H)$ is $6 \cdot 5 = 30$. \square

We can now apply Theorem 8.4 to calculate the defect d for Q_H . The formula is $126 - 120 + d = 30$, from which it follows that $d = 24$.

Corollary 8.6 *The small resolutions \widehat{Q}_H of the quintic threefolds Q_H have the following betti and Hodge numbers:*

$$\begin{aligned} b_2(\widehat{Q}_H) &= 25, & h^{1,1} &= 25, \\ b_3(\widehat{Q}_H) &= 12 = 2 + 2h^{2,1}, & h^{2,1} &= 5, & e &= 2h^{1,1} - 2h^{2,1} = 40. \end{aligned}$$

Proof: Insertion of $d = 24$ in (124). \square

In the well known manner for Calabi-Yau threefolds the isomorphism $H^2(V, \Omega^1) \cong H^1(V, \Theta)$ identifies the Hodge space $H^{2,1}$ with the space of infinitesimal deformations of V , $H^1(V, \Theta)$. This is by the above five-dimensional, hence the moduli space of these 120 nodal quintics (a Zariski open subset of $(\mathbb{P}^5)^\vee$) is also a global space of complex deformations of the small resolution. We can describe the space $H^{2,1}$ more concretely as follows. Consider the space $\mathcal{P}_5(Q_H)$; let $\mathcal{J} \subset \mathcal{P}_5(Q_H)$ be the subspace generated by the Jacobi ideal of Q_H ; since Q_H has five partial derivatives, \mathcal{J} is $5 \cdot 5 = 25$ dimensional, and \mathcal{J} cannot contribute to infinitesimal deformations, so we have

$$H^{2,1}(\widehat{Q}_H) \cong \mathcal{P}_5(Q_H)/\mathcal{J}.$$

As a final remark consider the Picard group $\text{Pic}(\widehat{Q}_H)$ and the orthocomplement of the hyperplane section $\text{Pic}^0(\widehat{Q}_H)$. Then $rk_{\mathbb{Z}}\text{Pic}^0(\widehat{Q}_H) = 24$, and the 45 \mathbb{P}^2 's give us privileged representatives in Pic^0 ; the 27 hyperplanes represent relations, so we have an exact sequence

$$\mathbb{Z}^{27} \longrightarrow \mathbb{Z}^{45} \longrightarrow \text{Pic}^0(\widehat{Q}_H) \longrightarrow 1,$$

and the kernel is six-dimensional. The sum sequence is then

$$1 \longrightarrow \mathbb{Z}^6 \longrightarrow \mathbb{Z}^{27} \longrightarrow \mathbb{Z}^{45} \longrightarrow \mathbb{Z}^{24} \longrightarrow 1, \quad (125)$$

and this is really dual to the sequence (4) for cubic surfaces.

Remark 8.7 The period map for this five-dimensional family of Calabi-Yau threefolds maps to the domain $\mathcal{D} = Sp(6, \mathbb{R})/U(1) \times U(5)$. Note that any hyperplane passing through one of the 45 \mathbb{P}^3 's will intersect \mathcal{I}_5 in the union of that \mathbb{P}^3 and a residual quartic; clearly these constitute the set of cusps for the period map, i.e., on the 45 lines in $(\mathbb{P}^5)^\vee$ (the dual \mathbb{P}^5) which parameterise the set of hyperplanes passing through one of the 45 \mathbb{P}^3 's, the period map maps to the boundary of the domain \mathcal{D} above. These 45 one-dimensional cusps meet in 27 points, i.e., zero-dimensional cusps, which correspond to the 27 hyperplane sections which split into the union of five \mathbb{P}^3 's. But we can say more. Noting that, excepting the hyperplanes above, all hyperplane sections of \mathcal{I}_5 are irreducible quintics, the worst that can happen is that the hyperplane passes through one of the 36 triple points of \mathcal{I}_5 . We will see below that these are still Calabi-Yau (Proposition 9.8), hence *not* contained in the boundary.

8.4 Tangent hyperplane sections

We now consider the case of a hyperplane tangent to \mathcal{I}_5 at a point $p \in \mathcal{I}_5$. In this case the section Q_p acquires an additional node. Note that the 121 nodes fall into two ‘‘orbits’’, one set of 120 on which $W(E_6)$ acts as a permutation group, and the additional point p . For a 121-nodal quintic the same calculation as above gives $e(Q_p) = 42$, $h^{2,1} = 4$, $h^{1,1} = 25$. It follows that the $H_4(Q_p, \mathbb{Q})$ is the same as for Q_x , $x \in \mathbb{P}^5$ generic. The difference to the generic case is in H_3 , more precisely in $H^{2,1}$. Indeed, we now require $\mathcal{P}_5(Q_p)$, that is, quintics through all 121 nodes, so as opposed to the general case, we now only have, for each of the five quartics in the Jacobi ideal of Q_p , since each contains p , a five-dimensional family of quintics, as above. But for the quartics through the 120 nodes which are *not* in the Jacobi ideal, we must take a hyperplane *through the point p* , so

Proposition 8.8 For a 121-nodal quintic Q_p , $p \in \mathcal{I}_5$, we have $\dim \mathcal{P}_5(Q_p) = 5 \cdot 5 + 1 \cdot 4 = 29$. \square

We can now apply Theorem 8.4 to calculate the defect:

$$d = 29 - 126 + 121 = 24.$$

Corollary 8.9 The betti numbers for the small resolution \widehat{Q}_p are

$$\begin{aligned} b_2(\widehat{Q}_p) &= 25, & h^{1,1} &= 25, \\ b_3(\widehat{Q}_p) &= 10, & h^{2,1} &= 4, & e &= 42. \end{aligned}$$

We remark that since $h^{1,1}$ is still 25, the sequence (125) still holds for Q_p .

9 Birational maps and the projection from a triple point

9.1 The cuspidal model

First we recall the notations $\mathcal{I}_5^{(1)}$ for the blow up of \mathcal{I}_5 at the 36 triple points, $\widetilde{\mathcal{I}}_5$ for the big resolution of \mathcal{I}_5 , and $\mathcal{I}_5^{(s)}$ for the small resolution. Note that on $\mathcal{I}_5^{(1)}$, each of the 120 lines has normal bundle $\mathcal{O}(-2)^{\oplus 3}$, hence each line can be blown down to an isolated singular point.

Definition 9.1 Consider the following birational transformation of \mathcal{I}_5 :

- i) Blow up the 36 triple points, $\varrho^{(1)} : \mathcal{I}_5^{(1)} \longrightarrow \mathcal{I}_5$;
- ii) Blow down the proper transforms of the 120 lines to 120 isolated singularities, $\varrho^{(2)} : \mathcal{I}_5^{(1)} \longrightarrow \widehat{\mathcal{I}}_5$.

Step ii) defines the *cuspidal model* $\widehat{\mathcal{I}}_5$.

This is a four-dimensional analogue of $\widehat{\mathcal{N}}_5$ of (73). Indeed, for each of the 36 hyperplane sections of Proposition 8.2, the proper transform on $\widehat{\mathcal{I}}_5$ is isomorphic to $\widehat{\mathcal{N}}_5$:

Lemma 9.2 Let $T \cong \mathcal{N}_5$ be one of the 36 special hyperplane sections of (123), and let \widehat{T} denote its proper transform on $\widehat{\mathcal{I}}_5$. Then $\widehat{T} \cong \widehat{\mathcal{N}}_5$.

Proof: Just check that the steps i) and ii) of Definition 9.1, when restricted to T , coincide with those of (73). \square

Let us mention that $\widehat{\mathcal{I}}_5$ “looks like” a ball quotient too, at least assuming a positive answer to Problem 4.19. We explain what “looks like” means in the following items.

- I1 Each isolated singularity is resolved by a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; the arrangement induced in each by the proper transforms of the 36 hyperplanes and 36 exceptional divisors is a *product*, consisting of three fibres in each fibering (i.e., $\{3 \text{ points}\} \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \{3 \text{ points}\} \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \{3 \text{ points}\}$, see 6.5.3). Hence this can be covered in an equivariant way by $E_\rho \times E_\rho \times E_\rho$ (see Lemma 2.5.2).
- I2 The proper transforms of the 36 hyperplane sections of Proposition 8.2 are by Lemma 9.2 isomorphic to $\widehat{\mathcal{N}}_5$, so, if the Problem 4.19 has an affirmative solution, these are ball quotients, with cusps being those isolated singularities of $\widehat{\mathcal{I}}_5$ which are contained in the given \widehat{T} .
- I3 Consider the 45 \mathbb{P}^3 's of Proposition 8.1. These are (the proper transforms of) the 45 \mathbb{P}^3 's of 6.5.8. These \mathbb{P}^3 's are also ball quotients, in fact in two different ways.
 - 1) There is a cover $Y \longrightarrow \mathbb{P}^3$, branched over the arrangement $\mathcal{A}(W(\mathbf{D}_4))$ in \mathbb{P}^3 , which is a ball quotient. This example can be derived from the solution 4) of (26) by means of the natural squaring map $m_2 : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$, $(x_0 : \dots : x_3) \mapsto (x_0^2 : \dots : x_3^2)$. Then the arrangement $\mathcal{A}(W(\mathbf{D}_4))$ is the pullback under m_2 of the six symmetry planes of the tetrahedron in the arrangement $\mathcal{A}(W(\mathbf{A}_4))$, and pulling back the solution 4), we get the cover $Y \longrightarrow \mathbb{P}^3$, branched along $\mathcal{A}(W(\mathbf{D}_4))$ (with branching degree 3 at each hyperplane), which is a ball quotient by a fix point free group.

- 2) There is a cover $Z \rightarrow \mathbb{P}^3$, branched along the arrangement $\mathcal{A}(W(\mathbf{F}_4))$ in \mathbb{P}^3 (but not a Fermat cover), which is a ball quotient; this example is explained in [H], Thm. 7.6.5, and is the *only* known ball quotient related to a plane arrangement in \mathbb{P}^4 which does *not* derive from those given by solutions of the hypergeometric differential equation.

Both of the arrangements mentioned, $\mathcal{A}(W(\mathbf{D}_4))$ and $\mathcal{A}(W(\mathbf{F}_4))$, arise naturally on the 45 \mathbb{P}^3 's: the first is the intersection with the 27 hyperplanes, the second is the intersection with the 36 hyperplanes.

9.2 Projection from a triple point

Let $p \in \mathcal{I}_5$ be one of the 36 triple points, and let h_p be the dual hyperplane (one of the 36 \mathbb{P}^4 's of 6.5.1). The projection of \mathbb{P}^5 from p is defined as follows. Consider the \mathbb{P}^4 of all lines through p ; this is just the dual h_p , and each line l_p through p corresponds to a unique point of h_p (its intersection with h_p). Since any point x of \mathbb{P}^5 is on a unique line $(l_x)_p$ through p , the map

$$\begin{aligned} \pi_p : \mathbb{P}^5 &\longrightarrow h_p \\ x &\longmapsto (l_x)_p \cap h_p \end{aligned} \tag{126}$$

gives the *projection from p* . Restricting to \mathcal{I}_5 this gives a generically finite (rational) map, which we also denote by $\pi_p, \pi_p : \mathcal{I}_5 \dashrightarrow h_p$.

Lemma 9.3 $\pi_p : \mathcal{I}_5 \dashrightarrow h_p$ is generically a double cover.

Proof: Since the triple point has multiplicity 3, a generic line will meet \mathcal{I}_5 in $(5 - 3) = 2$ further points. \square

Lemma 9.4 $\pi_p : \mathcal{I}_5 \dashrightarrow h_p$ is a quotient map by the group $G_p \cong \mathbb{Z}/2\mathbb{Z}$ generated by the reflection σ_p on the root p .

Proof: The reflection σ_p fixes h_p ; it is the inversion $((z_0 : z_1) \mapsto (z_1 : z_0))$ on any line l_p through p , where the homogenous coordinates are chosen such that $l_p \cap h_p = (1 : 1)$. Since \mathcal{I}_5 is mapped by σ_p onto itself, it follows that two points of $\mathcal{I}_5 \cap l_p$ are related by inversion on l_p . So the group action is manifest. \square

We now describe how to make π_p into a *morphism*. First of all, one must blow up p ; let $\varrho_p : \mathcal{I}_{5,p} \rightarrow \mathcal{I}_5$ denote this blow up. Let $(\mathcal{S}_3)_p$ be the copy of \mathcal{S}_3 which is the exceptional divisor at p . For any $x \in (\mathcal{S}_3)_p$, the line $(l_x)_p$ through p and intersecting h_p in the Segre cubic there, is tangent to \mathcal{I}_5 at the triple point p . Secondly, certain subvarieties get *blown down*. Indeed, suppose $(l_x)_p$ is contained in \mathcal{I}_5 for some $x \in \mathcal{I}_5$. Then, clearly, $(l_x)_p \mapsto (l_x)_p \cap h_p$, the whole line maps to a point, or in other words, gets blown down.

Lemma 9.5 The projection $\pi_p : \mathcal{I}_5 \dashrightarrow h_p$, which is well-defined on $\mathcal{I}_{5,p}$, blows down all linear subspaces on \mathcal{I}_5 which pass through p , and is a double cover outside the union \mathcal{L}_p of all such linear subspaces on \mathcal{I}_5 passing through p . \square

We now describe \mathcal{L}_p . Recall that the linear subspaces on \mathcal{I}_5 are the 45 \mathbb{P}^3 's and their intersections. Hence \mathcal{L}_p consists of all the \mathbb{P}^3 's and their intersections, which pass through p . Recall from 6.5.6 that this is the set of 15 of the 45 \mathbb{P}^3 's of Proposition 8.1. Therefore, we get

Lemma 9.6 The projection $\pi_p, p : \mathcal{I}_{5,p} \rightarrow h_p$ blows down the union of 15 \mathbb{P}^3 's to the 15 planes in h_p which are the intersection of \mathcal{S}_3 and \mathcal{N}_5 .

Now let $X = \mathcal{I}_{5,p}^\%$, the double cover of h_p branched along the union of \mathcal{S}_3 and \mathcal{N}_5 (which is of degree 8, so a double cover exists). X is clearly *singular along* the 15 planes. Indeed:

Lemma 9.7 $\pi_p, p : \mathcal{I}_{5,p} \rightarrow h_p$ factors over $\mathcal{I}_{5,p}^\%$, and $\Pi : X \rightarrow h_p$ is the double cover of \mathbb{P}^4 branched along the union $\mathcal{S}_3 \cup \mathcal{N}_5$.

Proof: This follows from the discussion above; the branch locus \mathcal{R} is the set:

$$\mathcal{R} = \{x \in \mathcal{I}_{5,p} \mid (l_x)_p \text{ is tangent to } \mathcal{I}_5 \text{ at } x\}.$$

This happens if either

- i) $x \in h_p$, since then x is fixed by σ_p ;
- ii) $x \in (\mathcal{S}_3)_p$, the exceptional divisor over p .

Therefore $\mathcal{R} = \mathcal{S}_3 \cup \mathcal{N}_5$. By Lemmas 9.5 and 9.6, 15 \mathbb{P}^3 's are blown down to \mathbb{P}^2 's, and outside of this locus, Π is 2:1. \square

9.3 Double octics and quintic hypersurfaces

With the result of Lemma 9.7 at hand, we can get a new slant on the quintic threefolds which are hyperplane sections of \mathcal{I}_5 . For this, consider a hyperplane section of the cover $\Pi : X \rightarrow h_p$, that is, let $H \subset h_p$ be a hyperplane, and let X_H be its inverse image in X :

$$\Pi_H : X_H \rightarrow H,$$

a double cover of \mathbb{P}^3 . The branch locus is $H \cap (\mathcal{S}_3 \cup \mathcal{N}_5)$, which is the union of a cubic and a quintic surface in \mathbb{P}^3 . Note the $H \cap \{\text{one of the 15 } \mathbb{P}^2\text{'s} \subset \mathcal{S}_3 \cap \mathcal{N}_5\}$ is a *line*, contained in both $H \cap \mathcal{S}_3$ and in $H \cap \mathcal{N}_5$. In other words, $H \cap (\mathcal{S}_3 \cup \mathcal{N}_5) = S_H \cup Q_H$, where S_H is the cubic surface, Q_H is the quintic surface, and

$$S_H \cap Q_H = \{15 \text{ lines}\}.$$

Proposition 9.8 *Let $X_H = \Pi^{-1}(H)$, the double cover of \mathbb{P}^3 branched along $S_H \cup Q_H$. Then there is a canonical model \overline{X}_H of X_H which is Calabi-Yau.*

Proof: We know the resolution of X ; it is given by “inverting” the projection from the node, by blowing up along the 15 planes $\mathcal{S}_3 \cap \mathcal{N}_5$, yielding $\mathcal{I}_{5,p}$. Let \overline{X}_H be the proper transform of X_H in $\mathcal{I}_{5,p}$. Assuming H to be sufficiently general, \overline{X}_H clearly has canonical singularities (as $\mathcal{I}_{5,p}$ does), so we must only show that it is Calabi-Yau. We note, however, that \overline{X}_H is (the proper transform on $\mathcal{I}_{5,p}$ of) a hyperplane section of \mathcal{I}_5 ! This is because the degree is invariant under projection, hence under Π . But this is a hyperplane section of \mathcal{I}_5 through the triple point p . Hence the proper transform on \overline{X}_H of the exceptional divisor $(\mathcal{S}_3)_p$ is a hyperplane section of \mathcal{S}_3 , i.e., a (generically smooth) cubic surface. This singularity is known to be canonical, and \overline{X}_H is, just as a nodal quintic, canonically Calabi-Yau. \square

Corollary 9.9 *The family of hyperplane sections of \mathcal{I}_5 passing through one of the 36 triple points p is, via projection, a family of Calabi-Yau threefolds which are degenerations of double octics.*

It is natural to ask the meaning of this in terms of variations of Hodge structures. Recalling that a Type III degeneration of a K3 surface, corresponding to a zero-dimensional boundary component of the period domain, is one like a quartic degenerating into four planes, it is natural to ask

Question 9.10 *Is a double cover of \mathbb{P}^3 branched over the union of a cubic and a quintic a semistable degeneration of a double octic?*

Remark 9.11 There is a notion of “connecting” moduli spaces of CY threefolds by degenerations, and the Corollary 9.9 shows that the moduli space of quintic hypersurfaces in \mathbb{P}^4 and the moduli space of double octics are connected; the birational transformations which are required for such “connections” are given here by projection in projective space, very geometric.

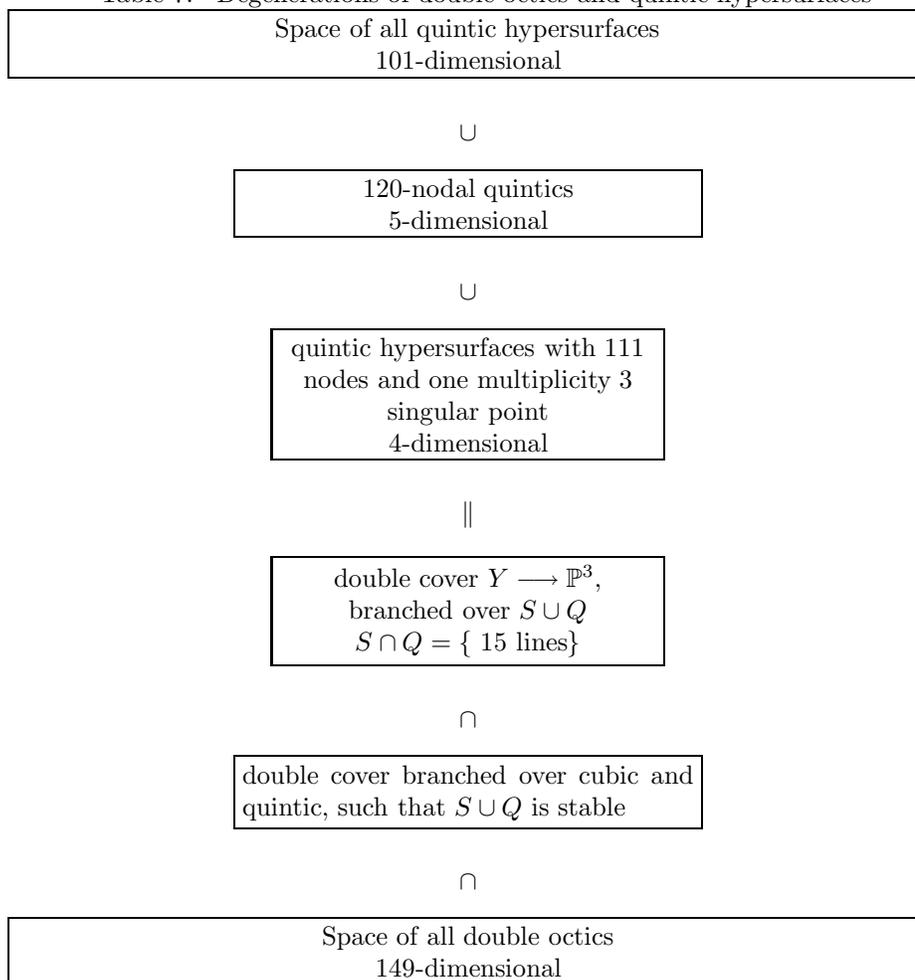
In Table 7 we give a rough description of these relations.

9.4 The dual picture

Now consider $\alpha(\mathcal{S}_3 \cup \mathcal{N}_5)$, with α the map (71) given by the quadrics on the ten nodes of \mathcal{S}_3 . By Theorem 3.12 and by definition of \mathcal{W}_{10} (72), we have

$$\alpha(\mathcal{S}_3 \cup \mathcal{N}_5) = \mathcal{I}_4 \cup \mathcal{W}_{10}, \tag{127}$$

Table 7: Degenerations of double octics and quintic hypersurfaces



and by Theorem 4.12, the intersection $\mathcal{I}_4 \cap \mathcal{W}_{10}$ consist of 10 quadric surfaces. Define \mathcal{W} to be the double cover of \mathbb{P}^4 branched along \mathcal{W}_{10} :

$$\tau : \mathcal{W} \longrightarrow \mathbb{P}^4 = (h_p)^\vee. \tag{128}$$

We may consider the fibre square:

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \pi \\ \tau : \mathcal{W} & \longrightarrow & \mathbb{P}^4 \end{array} \tag{129}$$

where $\pi : \mathcal{Y} \longrightarrow \mathbb{P}^4$ is defined in Definition 5.5. Then $\pi_{\mathcal{Z}} : \mathcal{Z} \longrightarrow \mathbb{P}^4$ is a Galois cover with Galois group $G_{\mathcal{Z}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $H \cong \mathbb{Z}/2\mathbb{Z} \subset G_{\mathcal{Z}}$ be the diagonal subgroup; it is a normal subgroup, and we may form the quotient

$$\eta : \mathcal{Z} \longrightarrow \mathcal{Z}', \quad \mathcal{Z}' = \mathcal{Z}/H.$$

Lemma 9.12 $\pi_{\mathcal{Z}} : \mathcal{Z} \longrightarrow \mathbb{P}^4$ factors over η , and $\eta' : \mathcal{Z}' \longrightarrow \mathbb{P}^4$ is a double cover, hence Galois.

Proof: This is a general fact about fibre squares of double covers like (129). □

Theorem 9.13 The rational map α induces a rational map of the double covers $\Pi : X \longrightarrow \mathbb{P}^4$ of Lemma 9.7 and $\eta' : \mathcal{Z}' \longrightarrow \mathbb{P}^4$ of Lemma 9.12. Furthermore, the rational map

$$\Xi : X \dashrightarrow \mathcal{Z}'$$

is Σ_6 -equivariant.

Proof: Recall from Lemma 4.9 that α blows up the ten nodes and blows down the tangent cones of the nodes to the quadric surfaces (on \mathcal{I}_4). $\Pi : X \longrightarrow h_p$ is a double cover branched along $\mathcal{S}_3 \cup \mathcal{N}_5$, and we can calculate the image of the branch locus under α . The ten nodes get blown up, the ten quadric cones (in \mathbb{P}^4) get blown down to quadric surfaces (in the exceptional \mathbb{P}^3 's). Let $\tilde{C} \subset X$ be the inverse image in X of the union of the ten quadric cones; then on $X \setminus \tilde{C}$, α is biregular. On the other hand, $\alpha(\tilde{C})$ is just the union of the ten quadric surfaces of the intersection $\mathcal{I}_4 \cap \mathcal{W}_{10}$. Consequently

$$\alpha|_{X \setminus \tilde{C}} : X \setminus \tilde{C} \longrightarrow \mathcal{Z}' \setminus (\eta')^{-1}(\mathcal{I}_4 \cap \mathcal{W}_{10})$$

is a regular morphism of double covers, and letting $C \subset h_p$ denote the ten quadric cones, $X \setminus \tilde{C} \longrightarrow h_p \setminus C$ is a double cover, as is also

$$\mathcal{Z}' \setminus (\eta')^{-1}(\mathcal{I}_4 \cap \mathcal{W}_{10}) \longrightarrow (h_p)^\vee \setminus \mathcal{I}_4 \cap \mathcal{W}_{10},$$

while $\alpha(C) = \mathcal{I}_4 \cap \mathcal{W}_{10}$. Hence in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Xi} & \mathcal{Z}' \\ \downarrow & & \downarrow \\ h_p & \xrightarrow{\alpha} & (h_p)^\vee \end{array}$$

Ξ is regular outside of \tilde{C} and maps \tilde{C} to $(\eta')^{-1}(\mathcal{I}_4 \cap \mathcal{W}_{10})$. Furthermore, everything is defined Σ_6 -equivariantly. This proves the Theorem. □

Corollary 9.14 \mathcal{I}_5 sits Σ_6 -equivariantly birationally in the center of the diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Y} \\ \downarrow & \searrow & \downarrow \\ & \mathcal{Z}' & \\ \downarrow & \searrow & \downarrow \\ \mathcal{W} & \longrightarrow & \mathbb{P}^4. \end{array}$$

This shows the relation between the quintic \mathcal{I}_5 and the Coble variety \mathcal{Y} .

Proof: We have the series of modifications

$$\begin{array}{ccccccc} \mathcal{I}_5 & \longrightarrow & \mathcal{I}_{5,p} & \xrightarrow{\beta} & X & \xrightarrow{\Xi} & \mathcal{Z}' \\ & & & & \downarrow & & \downarrow \\ & & & & \mathbb{P}^4 & \xrightarrow{\alpha} & \mathbb{P}^4, \end{array}$$

where $\mathcal{I}_5 \rightarrow \mathcal{I}_{5,p}$ blows up the node p , β blows down the 15 \mathbb{P}^3 's through the node to the 15 \mathbb{P}^2 's of the intersection $\mathcal{S}_3 \cap \mathcal{N}_5$, α and Ξ are as described above. Since \mathcal{Z}' clearly sits in the center of the diagram and all modifications are Σ_6 -equivariant, the Corollary follows. \square

10 \mathcal{I}_5 and cubic surfaces

10.1 The Picard group

Let $A_1(\mathcal{I}_5)$ be the Chow group of Weil divisors modulo algebraic equivalence. Clearly a generic hyperplane section yields an element in $A_1(\mathcal{I}_5)$, which we denote by n . Recall the reducible hyperplane sections of Proposition 8.1 which split each into the union of five copies of \mathbb{P}^3 . These subvarieties are divisors on \mathcal{I}_5 , hence also yield classes in the Chow group. These 45 divisors are related by 27 relations, the sum of the five classes in the Chow group being equivalent to n . Since \mathcal{I}_5 is normal, we have an injection $\text{Pic}(\mathcal{I}_5) \hookrightarrow A_1(\mathcal{I}_5)$. Let $\text{Pic}^0(\mathcal{I}_5)$ denote the orthogonal complement of the class n in $A_1(\mathcal{I}_5)$ with respect to this injection. Then we have

Lemma 10.1 *We have an exact sequence of \mathbb{Z} -modules,*

$$0 \longrightarrow \mathbb{Z}^6 \longrightarrow \mathbb{Z}^{27} \longrightarrow \mathbb{Z}^{45} \longrightarrow \text{Pic}^0(\mathcal{I}_5) \longrightarrow 0.$$

Proof: The 45 \mathbb{P}^3 's are classes in $A_1(\mathcal{I}_5)$ which generate $\text{Pic}^0(\mathcal{I}_5)$ (as they contain all singularities), and the 27 relations are those just mentioned, given by the 27 hyperplane sections. So the sequence is clear as soon as we have shown that the rank of $\text{Pic}^0(\mathcal{I}_5)$ is 24 (see the sequence (125)). This now follows from the Lefschetz hyperplane theorem, as the dimension of \mathcal{I}_5 is four, so there is an isomorphism between the H^2 's of \mathcal{I}_5 and a hyperplane section. We may apply the Lefschetz theorem because the singularities of \mathcal{I}_5 and of a hyperplane section are local complete intersections (see the book by Goresky & MacPherson for details). \square

Note that this sequence displays $\text{Pic}^0(\mathcal{I}_5)$ as an *irreducible* $W(E_6)$ -module. Furthermore, we see that just as in (125), this sequence is dual to the corresponding sequence for cubic surfaces.

10.2 \mathcal{I}_5 and cubic surfaces: combinatorics

We collect the facts relating the combinatorics of the 27 lines with those of \mathcal{I}_5 in Table 8.

11 The dual variety

Let \mathcal{I}_5^\vee be the projective dual variety to \mathcal{I}_5 ; since \mathcal{I}_5 is invariant under $W(E_6)$, so is \mathcal{I}_5^\vee . Although we do not have explicit equations for \mathcal{I}_5^\vee , we can say quite a bit about its geometry, just from the fact that it is dual to \mathcal{I}_5 .

11.1 Degree

First we show that *degree of $\mathcal{I}_5^\vee = 10m + 4k$* . Quite generally, one can say the following. Suppose we are given a variety $X \subset \mathbb{P}^n$ which has singular locus consisting of a set of *lines*, meeting each other in a set of *points*, and let us further assume that the situation is symmetric, i.e., each line contains the same number of points, each point being hit by the same number of lines; let us denote these numbers by $N = \# \text{ lines}$, $M = \# \text{ points}$, $\nu = \# \text{ points in each line}$ and $\mu = \# \text{ lines meeting at each point}$. Consider the dual variety X^\vee . We claim:

- There are N \mathbb{P}^{n-2} 's $\subset X^\vee$.
- Each \mathbb{P}^{n-2} is cut out by ν hyperplanes.

Table 8: Combinatorics of \mathcal{I}_5 and the 27 lines

Locus on a cubic surface (see Table 2)	Locus on \mathcal{I}_5
27 lines a_i, b_i, c_{ij}	27 hyperplane sections $\{a_i = 0\} \cap \mathcal{I}_5$, etc.
2 lines are skew	the hyperplanes intersect in one of 216 \mathbb{P}^3 's dual to the lines of 6.5.4; this \mathbb{P}^3 intersects \mathcal{I}_5 in the union of three planes and a quadric (see Lemma 4.6)
two lines are in a tritangent	the hyperplanes intersect in one of the 45 \mathbb{P}^3 's of 6.5.8
45 tritangents	the 45 \mathbb{P}^3 's of 6.5.8
Two tritangents meet in a line of the cubic surface	two of the 45 \mathbb{P}^3 's meet in a \mathbb{P}^2 ; this is one of the planes in the \mathbb{P}^3 defining the arrangement $\mathcal{A}(W(\mathbf{D}_4))$ as discussed in 6.5.8
Two tritangents meet in a line outside of the cubic surface	two of the 45 \mathbb{P}^3 's are <i>skew</i> , i.e., meet only in a line; this line is part of the singular locus of the arrangement $\mathcal{A}(W(\mathbf{D}_4))$ just mentioned
36 double sixes	36 triple points of \mathcal{I}_5 AND 36 copies of the Nieto quintic \mathcal{N}_5
Two double sixes are azygetic	two of the triple points lie on one of the 120 lines of the singular locus of \mathcal{I}_5
Two double sixes are syzygetic	two of the triple points do not lie on one of the 120 lines
A line is <i>not</i> contained in a double six	the hyperplane dual to the line contains the triple point which corresponds to the double six

- There are M such special hyperplane sections of X^\vee .
- Each of the M hyperplanes meets X^\vee in μ of the N \mathbb{P}^{n-2} 's.
- Hence, $\deg(X^\vee) = m\mu + rest$,

where the *rest* is given in terms of the local geometry around the given point. The proofs of these are immediate: each of the points corresponds to a hyperplane (=set of all hyperplanes through the point), each line defines dually a \mathbb{P}^{n-2} , and since X is singular along the line, each hyperplane through the line is *tangent* to X there \Rightarrow the dual $\mathbb{P}^{n-2} \subset X^\vee$. The other statements are then clear. To determine *rest*, consider the following. The set theoretic image of the given point in the dual variety is the *total* transform (*not* the proper transform) of the given point. This is set theoretically easy to compute, but there may be a multiplicity coming in.

We apply these considerations to \mathcal{I}_5 and \mathcal{I}_5^\vee : on \mathcal{I}_5 we have singular lines, $N=120$, $M=36$, $\nu=3$, $\mu=10$, and hence $\deg(\mathcal{I}_5^\vee) = m10 + rest$. In our case *rest* is easy to figure out: recall that we resolved the singularities of \mathcal{I}_5 by blowing up the 36 points, then the 120 lines; the resolving divisors over the points were copies of the Segre cubic. The variety dual to the Segre cubic is the Igusa quartic, and the image of the ten nodes on the Segre cubic are ten quadric surfaces (49) which are *tangent hyperplane sections*, i.e., the hyperplanes which meet the Igusa quartic in one such quadric and are tangent to it there. These ten hyperplanes are of course just the 10 \mathbb{P}^3 's on the dual variety being cut out by the chosen hyperplane section (see Proposition 11.1 below). This hyperplane section of \mathcal{I}_5^\vee may be *tangent* to \mathcal{I}_5^\vee along the Igusa quartic, hence

$$\deg(\mathcal{I}_5^\vee) = 10m + k \cdot 4.$$

11.2 Singular locus

Consider the 45 \mathbb{P}^3 's on \mathcal{I}_5 ; since there is a pencil of hyperplanes through each, the dual variety \mathcal{I}_5^\vee will have 45 singular lines, which meet in 27 points (which are dual to the 27 hyperplanes cutting out the 45 \mathbb{P}^3 's). These

27 points are of course A_i, B_i, C_{ij} . Applying our reasoning from above to this we see that $\deg(\mathcal{I}_5^{\vee\vee})=5+$ rest. We conclude rest=0, or in other words, *a resolution of singularities of \mathcal{I}_5^{\vee} is affected by blowing up the 45 lines simultaneously; there is no exceptional divisor over the 27 points.*

However, since we are dealing with fourfolds, \mathcal{I}_5^{\vee} could even be normal and still have a singular locus of dimension two. For example, it is reasonable to believe that the ten quadrics on each copy of the Igusa quartic \mathcal{I}_4 on the reducible hyperplane sections discussed below might be *singular* on \mathcal{I}_5^{\vee} , but that is of course just a guess. Furthermore, there is no reason whatsoever why the dual variety should be normal. In fact, it is a case of great exception when the dual variety is normal, the general case being that there is a singular parabolic divisor (coming from the intersection $\text{Hess}(X) \cap X$), as well as a double point locus, also (in general) a divisor, coming from the set of bitangents. In our case, however, since $\text{Hess}(\mathcal{I}_5) \cap \mathcal{I}_5$ consists of the union of the 45 \mathbb{P}^3 's, all of which get blown down, there is no parabolic *divisor*. But there is no easy way to exclude a double point divisor.

11.3 Reducible hyperplane sections

As already mentioned, \mathcal{I}_5^{\vee} contains 120 \mathbb{P}^3 's, each being cut out by three of the h 's, (in fact by a triple of azygetic double sixes), and each such intersection $h \cap \mathcal{I}_5^{\vee}$ consists of ten such \mathbb{P}^3 's, plus a copy of the Igusa quartic. There are 36 such hyperplane sections which decompose into ten \mathbb{P}^3 's and a copy of the Igusa quartic:

Proposition 11.1 *The 36 hyperplane sections $\mathbf{h} \cap \mathcal{I}_5$, $\mathbf{h} = h, h_{ij}, h_{ijk}$, are reducible, consisting of ten \mathbb{P}^3 's and a copy of the Igusa quartic \mathcal{I}_4 . The ten \mathbb{P}^3 's are just the K_{ijk} of (50), each a bitangent plane to \mathcal{I}_4 .*

Proof: These are the 36 hyperplanes dual to the 36 triple points of \mathcal{I}_5 ; at each such p ten of the 120 lines meet, and the triple point itself yields the copy of \mathcal{I}_4 (it is blown up with exceptional divisor $(\mathcal{S}_3)_p$, which is dual to $(\mathcal{I}_4)_p$, a copy of \mathcal{I}_4). \square

So restricted to the triple point, the duality $\mathcal{I}_5 \dashrightarrow \mathcal{I}_5^{\vee}$ yields precisely the dual map α of (71)!

The 120 \mathbb{P}^3 's meet two at a time in 270 \mathbb{P}^2 's, each of which is cut out by six of the h 's (2 triples of azygetic double sixes, two rows in a triple). Note that these 270 \mathbb{P}^2 's are the $t_6(2)$ of Table 5. Each \mathbb{P}^2 contains two nodes and five of the 27 points, as well as two of the 45 lines. Through each such line two of these \mathbb{P}^2 's pass (as each line is cut out by 12 of the h 's). Therefore in each h we have ten \mathbb{P}^3 's meeting in $\binom{10}{2} = 45$ \mathbb{P}^2 's which meet in 15 of the 45 \mathbb{P}^1 's, and contain 15 of the 27 points. The 15 lines and 15 points are just the singular locus of the Igusa quartic, and the ten \mathbb{P}^3 's are tangent to the Igusa quartic along quadrics, as mentioned earlier.

11.4 Special hyperplane sections

Inspection of the 27 forms and 27 points in \mathbb{P}^5 shows that each of the 27 hyperplanes contains *none* of the 27 points and *none* of the 45 lines; it follows that hyperplane sections such as $\mathcal{K} := \mathcal{I}_5^{\vee} \cap \{a_1 = 0\}$ are irreducible hypersurfaces in \mathbb{P}^4 with 45 isolated singularities, coming from the intersections with the singular lines of \mathcal{I}_5^{\vee} . As mentioned above, there may also be a singular locus coming from other singularities on \mathcal{I}_5^{\vee} . Furthermore, there are 40 \mathbb{P}^2 's lying on this threefold, and 16 hyperplanes in a_i which cut out ten of these on \mathcal{K} . The 16 hyperplanes are those 16 of the 216 \mathbb{P}^3 's which lie in a_i , corresponding to the 16 lines which a_i is skew to. The symmetry group of this threefold is $W(D_5)$. This is a *degeneration* of a *generic* hyperplane section, which will contain 120 \mathbb{P}^2 's.

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