The Surface Area of the Ellipsoid

With $a > b > 0$ the equation

$$\frac{x^2}{s^2 - a^2} + \frac{y^2}{s^2 - b^2} + \frac{z^2}{s^2} = 1 \tag{1}$$

for different values of the parameter s, represents three different families of confocal quadric surfaces: confocal ellipsoids for $s > a$, confocal hyperboloids of one sheet for $a > s > b$ and confocal hyperboloids of two sheets for $b > s > 0$. The three families of surfaces are mutually orthogonal. Taking s in the respective ranges one receives a set of orthogonal coordinates (ξ, η, ζ), which are refered to as *ellipsoidal coordinates*. They are related to the cartesian coordinates (x, y, z) through

$$x = \frac{\sqrt{(\xi^2 - a^2)(a^2 - \eta^2)(a^2 - \zeta^2)}}{a\sqrt{a^2 - b^2}} \tag{2}$$

$$y = \frac{\sqrt{(\xi^2 - b^2)(\eta^2 - b^2)(b^2 - \zeta^2)}}{b\sqrt{a^2 - b^2}} \tag{3}$$

$$z = \frac{\xi\eta\zeta}{ab}, \tag{4}$$

where $\xi \geq a \geq \eta \geq b \geq \zeta \geq 0$. In the following we choose $1 > a > b > 0$ and calculate the surface area of the ellipsoid $\xi = 1$. The surface areas of other ξ values follow from a simple scaling. For $\xi = 1$ we first consider the arc lengths in η and ζ direction. They are

$$ds_\eta^2 = \frac{(\eta^2 - \zeta^2)(1 - \eta^2)}{(\eta^2 - b^2)(a^2 - \eta^2)} d\eta^2 \tag{5}$$

for η and

$$ds_\zeta^2 = \frac{(\eta^2 - \zeta^2)(1 - \zeta^2)}{(b^2 - \zeta^2)(a^2 - \zeta^2)} d\zeta^2 \tag{6}$$

for ζ, respectively. Because of the orthogonality of the coordinates (η, ζ) the surface element in these coordinates reads

$$ds_\eta ds_\zeta = (\eta^2 - \zeta^2)\sqrt{\frac{(1 - \eta^2)}{(\eta^2 - b^2)(a^2 - \eta^2)}}\sqrt{\frac{(1 - \zeta^2)}{(b^2 - \zeta^2)(a^2 - \zeta^2)}} d\eta d\zeta. \tag{7}$$

For the surface area of the ellipsoid $\xi = 1$ we thus have

$$\begin{aligned} S = \quad & 8\int_b^a d\eta\, \eta^2 \sqrt{\frac{(1 - \eta^2)}{(\eta^2 - b^2)(a^2 - \eta^2)}} \int_0^b d\zeta \sqrt{\frac{(1 - \zeta^2)}{(b^2 - \zeta^2)(a^2 - \zeta^2)}} \\ - \; & 8\int_b^a d\eta \sqrt{\frac{(1 - \eta^2)}{(\eta^2 - b^2)(a^2 - \eta^2)}} \int_0^b d\zeta\, \zeta^2 \sqrt{\frac{(1 - \zeta^2)}{(b^2 - \zeta^2)(a^2 - \zeta^2)}}. \end{aligned} \tag{8}$$

The integrals are of elliptic type. For the η integrals one finds

$$\int_b^a d\eta \sqrt{\frac{(1 - \eta^2)}{(\eta^2 - b^2)(a^2 - \eta^2)}} = \frac{1}{a\sqrt{1 - b^2}} \left(\mathcal{K}(k_\eta) - b^2 \Pi(\chi_\eta, k_\eta)\right), \tag{9}$$

where $\mathcal{K}(k_\eta)$ is Legendre's complete elliptic integral of first kind with modulus

$$k_\eta^2 = \frac{a^2 - b^2}{a^2(1-b^2)} \tag{10}$$

and $\Pi(\chi_\eta, k_\eta)$ is Legendre's complete elliptic integral of third kind with parameter

$$\chi_\eta^2 = \frac{a^2 - b^2}{a^2} \tag{11}$$

and modulus k_η.

$$\begin{aligned} & \int_b^a d\eta\, \eta^2 \sqrt{\frac{(1-\eta^2)}{(\eta^2-b^2)(a^2-\eta^2)}} \\ = \quad & \frac{1}{2}\sqrt{1-b^2}a \left(\mathcal{E}(k_\eta) + \frac{b^2}{1-b^2}\mathcal{K}(k_\eta) + \frac{b^2(1-a^2)-b^4}{a^2(1-b^2)}\Pi(\chi_\eta, k_\eta) \right), \end{aligned} \tag{12}$$

where $\mathcal{E}(k_\eta)$ denotes Legendre's complete elliptic integral of second kind with modulus k_η. For the ζ integrals we get

$$\int_0^b d\zeta \sqrt{\frac{(1-\zeta^2)}{(b^2-\zeta^2)(a^2-\zeta^2)}} = \frac{1}{\sqrt{1-b^2}a}\Pi(\chi_\zeta, k_\zeta) \tag{13}$$

with modulus

$$k_\zeta^2 = 1 - k_\eta^2 = \frac{b^2(1-a^2)}{a^2(1-b^2)} \tag{14}$$

and parameter

$$\chi_\zeta^2 = \frac{-b^2}{1-b^2} \tag{15}$$

and

$$\begin{aligned} & \int_0^b d\zeta\, \zeta^2 \sqrt{\frac{(1-\zeta^2)}{(b^2-\zeta^2)(a^2-\zeta^2)}} \\ = \quad & -\frac{1}{2}\sqrt{1-b^2}a \left(\mathcal{E}(k_\zeta) - \frac{1}{a^2}\mathcal{K}(k_\zeta) + \frac{1-a^2-b^2}{a^2(1-b^2)}\Pi(\chi_\zeta, k_\zeta) \right). \end{aligned} \tag{16}$$

The surface area of the ellipsoid becomes

$$\begin{aligned} S \quad = \quad & 4\Pi(\chi_\zeta, k_\zeta) \left(\mathcal{E}(k_\eta) + \frac{b^2}{1-b^2}\mathcal{K}(k_\eta) + \frac{b^2(1-a^2)-b^4}{a^2(1-b^2)}\Pi(\chi_\eta, k_\eta) \right) \\ + \quad & 4\left(\mathcal{K}(k_\eta) - b^2\Pi(\chi_\eta, k_\eta)\right) \left(\mathcal{E}(k_\zeta) - \frac{1}{a^2}\mathcal{K}(k_\zeta) + \frac{1-a^2-b^2}{a^2(1-b^2)}\Pi(\chi_\zeta, k_\zeta) \right). \end{aligned} \tag{17}$$

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